

# APPROXIMATELY FINITELY ACTING OPERATOR ALGEBRAS

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ABSTRACT. Let  $E$  be an operator algebra on a Hilbert space with finite-dimensional  $C^*$ -algebra  $C^*(E)$ . A classification is given of the locally finite algebras  $A_0 = \text{alg } \varinjlim (A_k, \phi_k)$  and the operator algebras  $A = \varinjlim (A_k, \phi_k)$  obtained as limits of direct sums of matrix algebras over  $E$  with respect to star-extendible homomorphisms. The invariants in the algebraic case consist of an additive semigroup, with scale, which is a right module for the semiring  $V_E = \text{Hom}_u(E \otimes \mathcal{K}, E \otimes \mathcal{K})$  of unitary equivalence classes of star-extendible homomorphisms. This semigroup is referred to as the dimension module invariant. In the operator algebra case the invariants consist of a metrized additive semigroup with scale and a contractive right module  $V_E$ -action. Subcategories of algebras determined by restricted classes of embeddings, such as 1-decomposable embeddings between digraph algebras, are also classified in terms of simplified dimension modules.

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## 1. INTRODUCTION

Approximately finite (AF)  $C^*$ -algebras are classified in terms of the scaled  $K_0$ -group (Elliott [16]). This perspective subsumed earlier special cases of Glimm [19], Dixmier [10] and Bratteli [3] and marked the advent of  $K$ -theory in operator algebra. For general (non-selfadjoint) operator algebras of an approximately finite-dimensional nature the situation is more problematic and classification schemes have usually been restricted to those limit algebras  $A = \varinjlim A_k$  which have intrinsic coordinates in the form of a well-defined semi-groupoid. See, for example, Poon and Wagner [37], Hopenwasser and Power [24], Muhly and Solel [33], and Power [38], [39]. In this case the building block algebras  $A_k$  are poset algebras, the inclusions  $A_k \rightarrow A_{k+1}$  are regular (that is, decomposable into multiplicity one embeddings) and the algebras  $A$  are triangular (in the sense of Kadison and Singer [28]).

In the  $C^*$ -algebra direction there have been significant developments in the last ten years in the classification of amenable  $C^*$ -algebras using  $K$ -theory invariants. This is generally referred to as the Elliott programme; see, for example Elliott [17] and Dădărlat and Eilers [7]. At the same time for nonselfadjoint operator algebras there have been developments arising from viewpoints in ring theory, representation theory and the resolution of modules, as can be seen in Blecher, Muhly and Paulsen [2], Muhly and Solel [34] and Muhly [32], for example. In the present paper we generalise the basic model for  $C^*$ -algebra classification by involving representation and embedding theory for finite-dimensional operator algebras. This leads to classifications of nonselfadjoint approximately finite operator algebras in terms of what we call dimension module invariants.

We consider approximately finitely acting operator algebras as those separable algebras whose building block algebras are finitely acting in the sense of having finite-dimensional generated  $C^*$ -algebras. (This gives the operator algebra category AFA.) A fundamental setting occurs when these subalgebras are direct sums of matrix algebras over a single finitely acting operator algebra  $E$ . In particular the operator algebras determined by stationary systems are of this form. Here the template algebra  $E$  is quite general, notably it need not be a normed poset algebra (digraph algebra) and the connecting homomorphisms considered are general star-extendible homomorphisms. An essential point of departure with the self-adjoint theory is that these embeddings may not be decomposable in terms of multiplicity one embeddings.

Of particular interest are those template operator algebras which are either of finite embedding type or (in some sense) of tame embedding type. The former situation leads to a classification theory in parallel with Elliott's original  $K_0$  classification of AF  $C^*$ -algebras. Away from this discrete situation it is not the case that close embeddings are inner unitarily equivalent but we accommodate for this and obtain complete invariants by endowing the algebraic invariants with an appropriate metric space structure.

The main results provide a framework for the classification of specific families of AF operator algebras in terms of reduced dedicated invariants and we give a number of applications in this direction. For example we classify the operator algebras which are direct limits of direct sums of  $T_r$ -algebras and we consider the subcategory determined by the 1-decomposable (regular) embeddings. Also, although the usual operator algebra realisation of  $T_r$ , the upper triangular  $r$  by  $r$  matrices, has infinite embedding rank for  $r \geq 3$  there are natural finitely acting realisations of  $T_r$  of finite embedding rank, and in particular this is so for the inflation operator algebra  $T_r^{max}$  formed by inflating over the representations from semi-invariant projections. We determine the number of classes of indecomposable embeddings (the embedding rank) from  $T_r^{max}$  into the stable algebra  $T_r^{max} \otimes \mathcal{K}$  as

$$d(T_r^{max}) = \binom{2r+1}{r+1} - (r+1).$$

As a result we find that the classifying dimension module invariants for limits of matrix algebras over  $T_4^{max}$  (for example) is an additive semigroup of the form  $\varinjlim \mathbb{Z}_+^{121}$  together with a scale and a right action by a finite multiplicative semigroup.

There are a number of other motivations for obtaining nonselfadjoint generalisations of Elliott's fundamental theorem. We show for example how the abstract classification scheme resolves affirmatively perturbation problems of the type "Are close approximately finite operator algebras isomorphic?" For example we prove that if limits of matricial  $T_r$ -algebras (for fixed  $r$ ) are star-extendibly close then they are star-extendibly isomorphic. It appears to be a longstanding open question whether, in general, close separable operator algebras are isomorphic. (Examples of Choi and Christensen [4] show that separability is necessary.)

The general topic of perturbation and stability for operator algebras, originated by Kadison and Kastler [27], has been well-developed for  $C^*$ -algebras by Loring and many others. (See Loring [31].) In the nonselfadjoint direction a norm perturbation theory for

reflexive operator algebras can be traced in Choi and Davidson [5], Davidson [8], Lance [29] and Pitts [36]. On the other hand the study of stability for (star-extendible) inclusions of nonselfadjoint building block algebras, even in the case of digraph algebras, is less advanced and yet highly significant for the local structure and local characterisation of operator algebras. For example the family of finitely acting operator algebras does not have the perturbational stability property of Definition 6.1 below and so it is of interest to identify subfamilies that do. Here we shall make use of the recent result of Haworth [21] that the family of matricial  $T_r$ -algebras is stable in this sense.

Further motivation for identifying invariants for limit algebras  $A$  comes from the purely self-adjoint issue of the classification of  $C^*$ -subalgebra positions. That this connection can be made is due to the fact that one can often recover the pair  $\{A, A^*\}$  from the subalgebra position

$$A \cap A^* \subseteq C^*(A)$$

as a distinguished pair in the lattice of intermediate closed algebras. Thus invariants for  $A$  provide invariants for the position of  $A \cap A^*$  up to  $C^*$ -algebra automorphism of  $C^*(A)$ . (See [44] and Section 11.)

Our starting point is the consideration of a finitely acting operator algebra  $E$  and a family  $\mathcal{F}$  of star-extendible homomorphisms  $\phi : E \otimes M_n \rightarrow E \otimes M_m$  (quantum symmetries). We associate with  $\mathcal{F}$  the category  $\text{Lim } \tilde{\mathcal{F}}$  of operator algebras of direct systems whose partial embeddings belong to  $\mathcal{F}$ . Assuming natural algebraic and analytic closure properties for  $\mathcal{F}$ , and a certain functorial closure property, we classify the algebras  $A$  in  $\text{Lim } \tilde{\mathcal{F}}$  in terms of invariants determined by  $\mathcal{F}$ . These invariants consist of an ordered abelian group  $(G_{\mathcal{F}}(A), V_{\mathcal{F}}(A))$  together with a scale  $\Sigma_{\mathcal{F}}(A)$ , a metric  $d_A$  on the cone  $V_{\mathcal{F}}(A)$  and the action of a metrized semiring  $V_{\mathcal{F}}$  on this cone, where  $V_{\mathcal{F}}$  is determined solely by  $\mathcal{F}$ . In fact the additive semigroup  $V_{\mathcal{F}}(A)$ , with its metric and  $V_{\mathcal{F}}$ -module action, is the primary invariant which we refer to as the metrized dimension module of  $A$ . ( $K_0$  is known also as the dimension group in the case of AF  $C^*$ -algebras.) The group  $G_{\mathcal{F}}(A)$  is simply the enveloping Grothendieck group of  $V_{\mathcal{F}}(A)$ . For certain discrete families (in the sense of Definition 3.1) one may dispense with the metric space structure. In general the semiring product reflects the structure of compositions of embeddings between the building block algebras and the metric measures the distance between the inner unitary equivalence classes

of embeddings. In many examples the semiring  $V_{\mathcal{F}}$  is identifiable as a semigroup semiring  $\mathbb{Z}_+[S]$  for a semigroup  $S$  and the  $V_{\mathcal{F}}$ -module action reduces to an  $S$ -action.

For a perturbationally stable algebra  $E$  the invariants for the family  $\mathcal{F}_E$  of unrestricted star-extendible embeddings can be given more simply in the direct form

$$V_{\mathcal{F}_E}(A) = \text{Hom}_a(E, A \otimes \mathcal{K}), \quad \Sigma_{\mathcal{F}_E}(A) = \text{Hom}_a(E, A),$$

where  $\text{Hom}_a(-, -)$  denotes the classes of star-extendible homomorphisms, for approximate inner unitary equivalence, together with the natural metric, and in this case

$$V_{\mathcal{F}_E} = \text{Hom}_a(E \otimes \mathcal{K}, E \otimes \mathcal{K})$$

with the natural right action on  $\text{Hom}_a(E, A \otimes \mathcal{K})$ . Although this manifestation connects with  $C^*$ -algebraic  $KK$ -theory, albeit with added metric, the primary reason for our formulation of the invariants was the identification of  $V_{\mathcal{F}}$ -action preservation as the key to the so-called existence step in the proof of the classification theorems of Sections 4 and 5. In earlier treatments of partly self-adjoint operator algebras, such as are indicated in [40], [11], [14], [15], [43], [25] and especially [45], the existence step was resolved in context specific ways in terms of an identification of sufficient additional invariants, such as homology and multiscales with which to augment  $K_0$ . The dimension module formulation here however is widely applicable.

For a proper subfamily  $\mathcal{F} \subset \mathcal{F}_E$  the dimension module  $V_{\mathcal{F}}(A_0)$  is defined in terms of a direct system for  $A_0$  whose embeddings are morphisms in  $\mathcal{F}$ . A fundamental issue which we address is whether the dimension module is an invariant for the algebra or, alternatively, depends on the direct system in an essential way. This is relevant to the classification of limits of digraph algebras with respect to 1-decomposable embeddings and to the classification of standard masas up to automorphism, including approximately inner automorphisms. We introduce a natural property for the family  $\mathcal{F}$ , which we refer to as functoriality, and this provides a sufficient (although not necessary) condition for the dimension module to be well-defined. It is shown that the 1-decomposable maps are functorial in the case  $E = T_3$  but are not functorial in the case of complete bipartite digraph algebras. These results provoke the following rather deep but natural problems, which have connections with subfactor theory in the bipartite case.

Problem 1. Determine those digraphs  $G$  for which the 1-decomposable digraph algebra maps  $A(G) \otimes M_n \rightarrow A(G) \otimes M_m$ , for all  $m, n$ , give a functorial family.

Problem 2. Determine the functorial completion of nonfunctorial families and their embedding semirings.

The paper is organised as follows. In Sections 2-5 we develop the abstract theory of classification by metrized dimension modules. (Examples 3.5 to 3.9 indicate a variety of finitely acting operator algebras.) In Section 6 we discuss perturbational stability for finitely acting algebras and limit algebras and obtain complete invariants for various approximately finite operator algebras. In Sections 7 and 8 we give two applications in the case of families  $\mathcal{F}_E$  of unrestricted embeddings including the classification of the operator algebra limits of inflation algebras. In Sections 9, 10 and 11 we discuss functoriality for constrained embeddings and in Section 11 we indicate the connections with the classification of  $C^*$ -subalgebra positions (as indicated above) and with subfactors.

The main results of this paper were presented in May 1999 at the 27th Canadian Operator Theory and Operator Algebras Symposium, in Charlottetown, PEI, and at the 19th Great Plains Operator Theory Symposium in Ames, Iowa.

## 2. APPROXIMATELY FINITELY ACTING OPERATOR ALGEBRAS

There does not appear to be an accepted notion of an approximately finite operator algebra acting on a complex Hilbert space but this has not prevented the study of many classes of operator algebras of this nature. We now give our preferred definitions and identify various subcategories and supercategories. For convenience we restrict the discussion to separable operator algebras.

Denote the category of separable (closed) operator algebras by OA, with the understanding that algebras in OA are either represented or, equivalently, are equipped with a prescribed  $C^*$ -algebra inclusion  $A \rightarrow C^*(A)$ . The morphisms of OA will be taken to be the most natural ones for subalgebras of  $C^*$ -algebras per se, namely the star-extendible homomorphisms. These are the algebra homomorphisms that are restrictions of  $C^*$ -algebra homomorphisms between the generated  $C^*$ -algebras. In particular, if an operator algebra  $A$  is a closed union of a chain of closed subalgebras  $A_k$ , then the inclusions  $A_k \rightarrow A_{k+1}$  are star-extendible and  $C^*(A)$  is the closed union of the  $C^*$ -algebras  $C^*(A_k)$ . It is elementary but significant to note that completely isometrically isomorphic operator algebras need not be star-extendibly isomorphic.

Denote by  $\text{OA}_n$  the subclass of algebras which can be represented, star-extendibly, on a Hilbert space of dimension no greater than  $n$ . We refer to these algebras as *finitely acting* since, of course, they differ from the finite dimensional algebras in  $\text{OA}$ . The approximately finitely acting operator algebras are defined here as those which are locally approximable by finitely acting subalgebras.

**Definition 2.1.** The subcategory AFA of approximately finitely acting operator algebras consists of those operator algebras  $A$  in  $\text{OA}$  such that for each  $\epsilon > 0$  and finite family  $a_1, \dots, a_n$  in  $A$  there exists an algebra  $B$  in  $\text{OA}_k$ , for some  $k$ , and a star-extendible embedding  $\phi : B \rightarrow A$  such that  $d(a_i, \phi(B)) \leq \epsilon$  for  $i = 1, \dots, n$ .

This definition is a local formulation which we can broaden further to identify an apparently wider class of algebras. Let us write  $M \subseteq_\epsilon N$  for subspaces  $M, N$  of an operator algebra if  $\text{dist}(m, N_1) \leq \epsilon$  for all  $m$  in  $M_1$ , where  $M_1, N_1$  are the unit balls of  $M, N$ .

**Definition 2.2.** The subcategory  $\text{AFA}_\epsilon$  of almost approximately finitely acting operator algebras consists of those operator algebras  $A$  in  $\text{OA}$  such that for each  $\eta > 0$  and  $a_1, \dots, a_n$  in  $A$  there exists an algebra  $B$  in  $\text{OA}_k$ , for some  $k$ , and a  $C^*$ -algebra embedding  $\phi : C^*(B) \rightarrow C^*(A)$  such that  $\phi(B) \subseteq_\eta A$  and  $d(a_i, \phi(B)) \leq \eta$  for  $i = 1, \dots, n$ .

On the other hand we write LIM for the subcategory of algebras in AFA which are direct limit algebras  $\varinjlim A_k$  where  $A_k \in \text{OA}_k$  and the embeddings  $A_k \rightarrow A_{k+1}$  are star-extendible for all  $k$ .

In an exactly similar way for a subfamily  $\mathcal{E}$  of finitely acting operator algebras define the classes

$$\text{Lim}(\mathcal{E}) \subseteq \text{AF}(\mathcal{E}) \subseteq \text{AF}(\mathcal{E})_\epsilon$$

where  $\text{Lim}(\mathcal{E})$  is the subclass of LIM consisting of limit algebras whose building block algebras lie in  $\mathcal{E}$ . If  $\mathcal{E}$  is the family of elementary  $E$ -algebras, by which we mean the finite-dimensional operator algebras of the form

$$E \otimes M_{n_1} \oplus \dots \oplus E \otimes M_{n_k},$$

then we write  $\text{AF}_E$  for  $\text{AF}(\mathcal{E})$ . In particular  $\text{AF}_\mathbb{C}$  is the class of AF  $C^*$ -algebras.

For the family  $\mathcal{E}$  of self-adjoint finite dimensional  $C^*$ -algebras it is essentially a well-known result of Glimm that these classes coincide [19]. The coincidence or otherwise of analogous nonself-adjoint categories is beginning to receive attention and we shall make use of Glimm type results in this direction. However we do not know if  $\text{LIM} = \text{AFA}$  or if  $\text{AFA} = \text{AFA}_e$ .

More specifically we shall be concerned with subcategories of  $\text{LIM}$  which derive from a given family  $\mathcal{F}$  of star-extendible embeddings  $\phi : A_1 \rightarrow A_2$  between finitely acting operator algebras. It is natural to require, as we do, the following properties of  $\mathcal{F}$ .

- (i)  $\mathcal{F}$  is closed under inner unitary conjugacy.
- (ii)  $\mathcal{F}$  is closed under compositions where such compositions are defined.
- (iii)  $\mathcal{F}$  is matricially stable: if  $\phi \in \mathcal{F}$  then the map  $\phi \otimes id : A_1 \otimes M_n \rightarrow A_2 \otimes M_n$  belongs to  $\mathcal{F}$ .
- (iv)  $\mathcal{F}$  is sum closed : if  $\phi, \psi \in \mathcal{F}$  with the same domain algebra  $A_1$  and range algebra  $A_2$ , then the map  $\phi \oplus \psi$  with domain  $A_1$  and range  $A_2 \otimes M_2$  belongs to  $\mathcal{F}$ .
- (v)  $\mathcal{F}$  is complete with respect to the norm topology.

We refer to a family  $\mathcal{F}$  satisfying the properties (i) to (v) simply as a *closed family* of maps.

**Definition 2.3.** Let  $\mathcal{F}$  be a family of star-extendible homomorphisms between finitely acting operator algebras which is a closed family in the sense above. Then  $\text{Lim } \mathcal{F}$  denotes the subset of algebras in  $\text{LIM}$  consisting of limit algebras of the form  $\varinjlim (A_k, \phi_k)$ , where  $\phi_k \in \mathcal{F}$ , for all  $k$ .

We also consider direct sums of building block algebras with admissible homomorphisms whose partial homomorphisms belong to  $\mathcal{F}$ . In this case we may define for a closed family  $\mathcal{F}$  the associated closed family  $\tilde{\mathcal{F}}$  of embeddings

$$\psi : E_1 \oplus \cdots \oplus E_k \rightarrow F_1 \oplus \cdots \oplus F_l$$

which have decompositions

$$\psi = \Sigma \oplus \psi_{ij}$$

where each map  $\psi_{ij} : E_i \rightarrow F_j$  belongs to  $\mathcal{F}$ . Basic examples here are the family  $\mathcal{F}_E$  of all star-extendible embeddings between the  $E$ -algebras (algebras of the form  $E \otimes M_n$ )



and the family  $\tilde{\mathcal{F}}_E$  of all star-extendible embeddings between the elementary  $E$ -algebras. More generally one may consider building block algebras that are Morita equivalent to  $E$  although we do not do so here.

**Remark 2.4** Let  $\mathcal{R}$  denote the closed family of maps between digraph algebras which are regular, or, equivalently, which are 1-decomposable. (See Example 3.7.) The most natural operator algebras in LIM are those belonging to  $\text{Lim}(\mathcal{R})$ . In particular, if  $A$  is a closed subalgebra of an AF  $C^*$ -algebra which contains a standard regular AF diagonal subalgebra then  $A$  belongs to  $\text{Lim}(\mathcal{R})$ . These are the algebras which have been most extensively studied, particularly in the triangular case. (See [41].)

Apart from the intrinsic interest in understanding other operator algebras in LIM there are additional reasons why it is wise to admit building block algebras which are more general than digraph operator algebras. Firstly, limits of elementary digraph algebras with respect to *nonstar*-extendible embeddings may yield operator algebras in LIM which are not locally approximable by digraph algebras. (See [23] for example.) Secondly, consider an operator algebra  $A$  of the form

$$A = \begin{bmatrix} C & S \\ 0 & C \end{bmatrix}$$

which acts on the Hilbert space  $H \oplus H$  where  $C$  is an abelian AF  $C^*$ -algebra and  $S$  is a  $C$ -bimodule in the algebra of compact operators, which contains no nonzero finite rank operators. Although  $A$  may belong to LIM, the only digraph subalgebras of  $A$  are abelian and so certainly  $A$  is not a star-extendible limit of digraph algebras. In examples such as these it is appropriate to consider what we shall refer to as inflation digraph algebras. These may be completely isomorphic to digraph algebras and yet not star-extendibly so.

### 3. EMBEDDING SEMIRINGS AND EMBEDDING RANK

We now define the metrized embedding semiring  $V_{\mathcal{F}}$  of a closed family of maps together with related terminology. We also give a Krull-Schmidt type theorem for maps between finitely acting operator algebras which shows in particular that the embedding semiring admits cancellation as an abelian semigroup. At the end of the section we consider various families of finitely acting operator algebras and their embedding semirings.

Let  $E$  be a finitely acting unital operator algebra. Write  $\text{Hom}(E \otimes \mathcal{K}, E \otimes \mathcal{K})$  for the family of star-extendible algebra homomorphisms  $\phi : E \otimes \mathcal{K} \rightarrow E \otimes \mathcal{K}$ . We generally refer to such homomorphisms simply as maps. Two maps  $\phi, \psi$  are said to be inner equivalent or, more emphatically, inner unitarily equivalent, if there is a unitary  $u$  in the unitisation of  $E \otimes \mathcal{K}$  for which  $\phi = (\text{Ad } u) \circ \psi$ . On identifying  $E$  with  $E \otimes p \subseteq E \otimes \mathcal{K}$ , with  $p$  a rank one projection, the map  $\phi$  restricts to define a map  $\phi_r : E \rightarrow E \otimes M_n$ . Conversely, each such map (a quantum symmetry in the sense of Ocneanu [35]) determines a unique map in  $\text{Hom}(E \otimes \mathcal{K}, E \otimes \mathcal{K})$ . With modest abuse of notation we write  $\phi$  for both of these maps.

Assume that  $\mathcal{F}$  is a closed family of maps, so that  $\mathcal{F}$  satisfies the properties (i) - (v) of the last section. Write  $V_{\mathcal{F}}$  for the set of inner unitary equivalence classes  $[\phi]$  of the induced maps on  $E \otimes \mathcal{K}$  (with the usual convention of taking unitaries in the unitisation of  $E \otimes \mathcal{K}$ ). Then  $V_{\mathcal{F}}$  is an additive abelian group with addition given by  $[\phi] + [\psi] = [\phi \oplus \psi]$ , and a multiplicative semigroup with product determined by composition;  $[\phi][\psi] = [\phi \circ \psi]$ . Also, multiplication is distributive over addition and  $V_{\mathcal{F}}$  is a semiring with zero element and multiplicative identity. Write  $V_E$  and also  $\text{Hom}_u(E \otimes \mathcal{K}, E \otimes \mathcal{K})$  to indicate the semiring determined by all maps. Then  $V_{\mathcal{F}}$  is a subsemiring of  $V_E$ . The enveloping ring  $R_{\mathcal{F}}$  of  $V_{\mathcal{F}}$  may be considered as the usual Grothendieck group of  $(V_{\mathcal{F}}, +)$  composed of formal differences together with the ring product given by

$$([\phi] - [\psi])([\nu] - [\mu]) = ([\phi][\nu] + [\psi][\mu]) - ([\psi][\nu] + [\phi][\mu]).$$

In view of Theorem 3.4 below the embedding semiring has additive cancellation and  $V_{\mathcal{F}}$  embeds injectively in the enveloping ring.

**Definition 3.1.** (i) The metrized semiring of the closed family  $\mathcal{F}$  is the semiring  $V_{\mathcal{F}}$  together with the metric  $d$  for which

$$d([\psi], [\phi]) = \inf_u \|\phi - (\text{Ad } u) \circ \psi\|$$

where the infimum is taken over the unitary group of the unitisation of  $E \otimes \mathcal{K}$ . We say that the family is discrete if this metric is discrete.

(ii) The embedding semiring  $V_E$  of the operator algebra  $E$  is the metrized semiring of the family of all star-extendible homomorphisms  $E \otimes \mathcal{K} \rightarrow E \otimes \mathcal{K}$ .

The terms in the following definitions have their counterparts in the representation theory of finite-dimensional complex algebras.

**Definition 3.2.** A (star-extendible) map between operator algebras is indecomposable if it cannot be written as a direct sum of two nonzero maps.

**Definition 3.3.** A unital operator algebra  $E$  is said to have finite embedding type if there are only finitely many (inner) unitary equivalence classes of indecomposable maps in  $\text{Hom}(E \otimes \mathcal{K}, E \otimes \mathcal{K})$ . The embedding rank  $d(E)$  is defined to be the number of these classes.

Finite embedding type means precisely that the embedding semiring

$$V_E = \text{Hom}_u(E \otimes \mathcal{K}, E \otimes \mathcal{K})$$

is finitely generated as an additive abelian semigroup.

Suppose now that  $C^*(E)$  is simple. Consider the multiplicity of  $\phi$ , denoted  $\mu(\phi)$ , to be the multiplicity of the star-extension  $\tilde{\phi} : C^*(A_1) \rightarrow C^*(A_2)$ . As we have remarked in the introduction, it is a basic point of departure with  $C^*$ -algebra theory that there are indecomposable maps of multiplicity greater than one. Let us say that the map  $\phi : A_1 \rightarrow A_2$  is  $k$ -decomposable if it admits a direct sum decomposition  $\phi = \phi_1 + \cdots + \phi_r$  where  $\mu(\phi_i) \leq k$  for  $1 \leq i \leq r$ . Also let us say that the finitely acting operator algebra  $E$  is  $k$ -decomposable if every map  $E \rightarrow E \otimes \mathcal{K}$  is  $k$ -decomposable and there are indecomposable maps of multiplicity  $k - 1$ . That  $E$  may be 1-decomposable without being of finite embedding type in the sense below is evident in the light of the elementary Toeplitz algebra  $L_2$  of Example 3.8.

When  $C^*(E)$  is simple the semiring  $V_{\mathcal{F}}$  with metric  $d$  may be viewed as both a graded manifold, graded by the multiplicity function  $\mu([\phi]) = \mu(\phi)$  and as a graded semiring. That is, there is a disjoint union

$$V_{\mathcal{F}} = \{0\} \sqcup V_{\mathcal{F}}^{(1)} \sqcup V_{\mathcal{F}}^{(2)} \sqcup \dots$$

where the subsets

$$V_{\mathcal{F}}^{(k)} = \{[\phi] : \mu([\phi]) = k\}, \quad k = 1, 2, \dots,$$

are open-closed and satisfy

$$V_{\mathcal{F}}^{(k)} + V_{\mathcal{F}}^{(l)} \subseteq V_{\mathcal{F}}^{(k+l)}, \quad \text{and} \quad V_{\mathcal{F}}^{(k)} V_{\mathcal{F}}^{(l)} \subseteq V_{\mathcal{F}}^{(kl)}.$$

Also the metric  $d$  is graded in the sense that if  $d([\phi], [\psi]) < 1$  then  $[\phi], [\psi]$  belong to the same subspace  $V_{\mathcal{F}}^{(k)}$ . The fact that  $V_{\mathcal{F}}^{(k)}$  is a locally Euclidean manifold is exploited in Section 6 in the consideration of the perturbational stability of algebras in  $\text{Lim}\mathcal{F}$ .

One can also bring into play the 2-sided module

$${}_F V_E = \text{Hom}_u(E \otimes \mathcal{K}, F \otimes \mathcal{K})$$

which is a right  $V_E$ -module and left  $V_F$ -module. This is relevant in the identification of the embedding semiring  $V_{E \oplus F}$  of the direct sum of two finitely acting operator algebras as the matricial semiring

$$\begin{bmatrix} V_E & {}_E V_F \\ {}_F V_E & V_F \end{bmatrix}.$$

However we shall not have cause here to consider such bimodules as we restrict attention to finitely acting operator algebras  $E$  which are indecomposable. We remark that the strict counterpart to the finite *representation* type of a complex algebra (see [18] for example) is that  ${}_C V_E$  have finite rank. But, as we note in the next paragraph, this is always the case in view of the star extendibility of the maps.

Recall that the Krull-Schmidt theorem for finite-dimensional complex algebra ensures that every finite-dimensional module admits a unique decomposition as a sum of finitely many indecomposable modules. The counterpart fact here, that every star-extendible representation  $\phi : E \rightarrow M_N$  admits, uniquely, an indecomposable decomposition  $\phi = \phi_1 \oplus \cdots \oplus \phi_r$  is elementary, at least if  $E$  is irreducible. For in this case the star extension  $\tilde{\phi} : C^*(E) \rightarrow M_N$ , which is given a priori, is a map  $M_r \rightarrow M_N$  which decomposes into a sum of multiplicity one embeddings. However we wish to show, more generally, that maps between two finitely acting operator algebras admit unique indecomposable decompositions, despite the fact that there may be uncountably many indecomposables and that the indecomposables need not have multiplicity one. Again, the proof is elementary  $C^*$ -algebra.

**Theorem 3.4. (Krull-Schmidt theorem)** *Let  $\phi : E_1 \rightarrow E_2$  be a unital map between finitely acting operator algebras. Then  $\phi$  admits a decomposition  $\phi = \phi_1 \oplus \cdots \oplus \phi_r$  into indecomposable maps and this decomposition is unique up to order and inner unitary equivalence.*

*Proof.* Let  $\{P_1, \dots, P_r\}$  be a maximal family of orthogonal projections in  $E_2 \cap E_2^*$  which reduces  $\phi(E_1)$ . Then  $\phi = \phi_1 \oplus \cdots \oplus \phi_r$ , where  $\phi_i(a) = P_i \phi(a)$ , and this is an indecomposable decomposition. Suppose that  $\psi = \psi_1 \oplus \cdots \oplus \psi_s$  is another indecomposable decomposition, and let  $Q_j = \psi_j(1)$ . Then the projections  $Q_1, \dots, Q_s$  belong to  $E_2 \cap E_2^*$  and reduce  $\phi(E_1)$ . It follows that the projections in the finite dimensional  $C^*$ -algebra  $C^*(P_i Q_j P_i)$  belong to  $E_2 \cap E_2^*$  and reduce  $\phi(E_1)$ . Thus, from the maximality of the family  $\{P_1, \dots, P_r\}$  it follows that for each pair  $i, j$  the positive operator  $P_i Q_j P_i$  is a scalar multiple of  $P_i$ , since otherwise the generated  $C^*$ -algebra contains a nonzero projection strictly less than  $P_i$ .

If  $\{Q_1, \dots, Q_s\}$  is also a maximal family then for each pair  $i, j$  we have  $Q_j P_i Q_j = \mu Q_j$  for some  $\mu$  (depending on the pair  $i, j$ ). It follows there is a permutation  $\pi$  such that  $P_i$  and  $Q_{\pi(j)}$  are unitarily equivalent in  $E_2 \cap E_2^*$  and the theorem follows.  $\square$

There is no extensive theory of finitely acting operator algebras which is ready-to-hand. Indeed, the problem of determining a general classification scheme is no less involved than that of classifying  $n$ -tuples of finite dimensional operators up to unitary equivalence. (Some related themes are indicated in Muhly [32].) We can largely bypass this issue here as our concern is directed at the role of the embedding semiring  $V_E$ , and its subsemirings, in the classification of limit algebras.

In the remainder of this section we consider a variety of natural template algebras  $E$ . In fact all these algebras can be viewed as locally partially isometric representations of the complex semigroup algebra of a finite semigroup.

**Example 3.5 The semiring  $V_{T_2}$ .** For  $r > 1$  let  $T_r$  denote the operator algebra subalgebra of  $M_r$  consisting of upper triangular matrices, so that  $T_r$  is spanned by the matrix units  $\{e_{i,j} : 1 \leq i \leq j \leq r\}$ . For  $r = 1$ ,  $T_r = \mathbb{C}$ ,  $V_{\mathbb{C}} = \mathbb{Z}_+$  and  $d(\mathbb{C}) = 1$ .

For  $r = 2$  we note that the maps  $T_2 \rightarrow T_2 \otimes M_n$  are 1-decomposable and that there are three classes of 1-decomposable embeddings. Thus  $d(T_2) = 3$  and as an additive semigroup  $V_{T_2} = \mathbb{Z}_+^3$ . Representations for the classes of indecomposables are given by the three maps

$\theta_i : T_2 \rightarrow T_2 \otimes M_2, 0 \leq i \leq 2$  given by

$$\theta_0 : \begin{bmatrix} x & y \\ & z \end{bmatrix} \rightarrow \left[ \begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & x & y & 0 \\ \hline & & z & 0 \\ & & 0 & 0 \end{array} \right],$$

$$\theta_1 : \begin{bmatrix} x & y \\ & z \end{bmatrix} \rightarrow \left[ \begin{array}{cc|cc} x & y & 0 & 0 \\ 0 & z & 0 & 0 \\ \hline & & 0 & 0 \\ & & 0 & 0 \end{array} \right], \quad \theta_2 : \begin{bmatrix} x & y \\ & z \end{bmatrix} \rightarrow \left[ \begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ & x & y & \\ & 0 & z & \end{array} \right],$$

To see this note that for a map  $\phi : T_2 \rightarrow T_2 \otimes M_n$  the partial isometry  $v = \phi(e_{12})$  has  $2 \times 2$  block upper triangular form and moreover (by star-extendibility) the projections  $v^*v$  and  $vv^*$  are block diagonal. Thus  $v = \begin{bmatrix} v_1 & v_2 \\ 0 & v_2 \end{bmatrix}$  where each  $v_i$  is also a partial isometry. (Such partial isometries, whose block entries are themselves partial isometries, are referred to as regular partial isometries with respect to the given block structure.) It follows that  $[\phi] = r_0[\theta_0] + r_1[\theta_1] + r_2[\theta_2]$  where  $r_0 = \text{rank } v_2$ ,  $r_1 = \text{rank } v_1$  and  $r_2 = \text{rank } v_3$ .

In  $V_{T_2}$  the three elements  $[\theta_i]$  form a semigroup  $S$  and  $V_{T_2}$  is the semigroup semiring  $\mathbb{Z}_+[S]$ .

The operator algebras of  $\text{Lim } \tilde{\mathcal{F}}_{T_2}$  were classified in [40] by augmenting the  $K_0$  invariants by a partial order on the scale of  $K_0$  which derives from partial isometries in the algebra. Also it follows from Heffernan [22] that

$$\text{Lim } (\tilde{\mathcal{F}}_{T_2}) = \text{AF}(\tilde{\mathcal{F}}_{T_2}) = \text{AF}(\tilde{\mathcal{F}}_{T_2})_\epsilon.$$

**Example 3.6 The semiring  $V_{T_3}$ .** The operator algebra  $T_3$  (acting on  $\mathbb{C}^3$  in the usual way) has infinite embedding type. To see this consider the maps  $\phi_\alpha : T_3 \rightarrow T_3 \otimes M_3$  given

by

$$\phi_\alpha : \begin{bmatrix} a & x & z \\ & b & y \\ & & c \end{bmatrix} \rightarrow \left[ \begin{array}{ccc|ccc|ccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a & 0 & x_1 & 0 & x_2 & z & 0 \\ \hline & & & a & x_4 & 0 & x_3 & 0 & z \\ & & & 0 & b & 0 & 0 & y_1 & y_2 \\ & & & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline & & & & & & b & y_4 & y_3 \\ & & & & & & 0 & c & 0 \\ & & & & & & 0 & 0 & c \end{array} \right]$$

with

$$\begin{bmatrix} x_1 & x_2 \\ x_4 & x_3 \end{bmatrix} = x \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}, \begin{bmatrix} y_1 & y_2 \\ y_4 & y_3 \end{bmatrix} = y \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}$$

where  $0 \leq \beta, \alpha \leq 1, \alpha^2 + \beta^2 = 1$ . If  $\alpha \neq 0, 1$  then  $\phi_\alpha$  is an indecomposable map with multiplicity 2. Indeed suppose that  $\phi' + \phi''$  is a nontrivial orthogonal direct sum decomposition. Then  $\phi'(e_{11}) = f_{33}$  or  $f_{44}$  and  $\phi'(e_{22}) = f_{55}$  or  $f_{77}$ , where  $(e_{ij})$  and  $(f_{ij})$  are the underlying matrix unit systems. Then

$$\|\phi'(e_{12})\| = \|\phi'(e_{11})\phi(e_{12})\phi'(e_{22})\| = \|f_{ii}\phi'(e_{12})f_{jj}\| = |\alpha| \text{ or } |\beta|$$

contrary to the fact that  $\phi'$  is isometric.

The maps  $\phi_\alpha \phi_\gamma$  are not inner conjugate if  $\alpha \neq \gamma$  since, for example, the norms of the block (1,2) entries of  $\phi_\alpha(e_{1,2})$  and  $\phi_\gamma(e_{1,2})$  differ.

On the other hand one can see that the block (1,2) entry of  $\phi_\alpha^{(n)}(a)$  has norm tending to zero, where  $a \in T_3$  and  $\phi_\alpha^{(n)}$  is the  $n$ -fold decomposition. From this it follows that in  $V_{T_3}$  the distance  $d([\phi_\alpha^{(n)}], [\phi_\beta^{(n)}])$  tends to zero as  $n$  tends to infinity. A consequence of this is that the stationary limit operator algebras determined by the maps  $\phi_\alpha$  for  $0 < \alpha < 1$  all coincide (whilst their algebraic limits do not).

**Example 3.7 Digraph algebras and regular maps.** A finite poset  $\mathcal{P} = \{v_1, \dots, v_n\}$  gives rise to a complex algebra  $A(\mathcal{P})$  with  $\mathbb{C}$  basis  $\{e_{ij}\}$  where  $i, j \in \{1, \dots, n\}$  and  $v_i \leq v_j$ . A natural realisation of  $A(\mathcal{P})$  as a finitely acting operator algebra arises when  $\{e_{ij}\}$  is taken to be a multiplicatively closed subset of a complete matrix unit system for  $M_n(\mathbb{C})$ . We refer

to such operator algebras as digraph algebras. Formally a digraph algebra  $A$  is a finitely acting operator algebra which contains a maximal abelian self-adjoint subalgebra (masa) of its generated  $C^*$ -algebra. The digraph  $G(A)$  of  $A$  is defined as the poset  $\mathcal{P}$ , viewed as a transitive reflexive directed graph with no multiple edges. The reduced digraph  $G^r(A)$  is the asymmetric digraph obtained from  $G(A)$  by identifying vertices in each maximal complete subgraph  $K_n \subseteq G(A)$ . In particular  $G(T_2) = G(T_2(\mathbb{C}) \otimes M_n(\mathbb{C}))$ , the digraph with two vertices, two loop edges and one proper edge.

Reciprocally, let us write  $A(G)$  for the digraph operator algebra determined by a digraph  $G$  with the properties above. Since the semirings  $V_{A(G)}$  and  $V_{A(G^r)}$  coincide we may confine attention to reduced (antisymmetric) digraphs.

Let  $G, H$  be connected and antisymmetric digraphs in the sense above. Then there are only finitely many equivalence classes of multiplicity one maps  $\phi : A(G) \rightarrow A(H) \otimes \mathcal{K}$ . Indeed each such class corresponds to a digraph homomorphism  $G \rightarrow H$ . Thus there is a natural multiplicative semigroup injection

$$\text{End}(G) \rightarrow \text{Hom}_u(A(G) \otimes \mathcal{K}, A(G) \otimes \mathcal{K}) = V_{A(G)}$$

and a semiring injection  $i : \mathbb{Z}_+[\text{End}(G)] \rightarrow V_{A(G)}$ , where  $\text{End}(G)$  indicates the endomorphism semigroup of  $G$ . In particular as we saw above the map  $i$  is a surjection if  $A(G) = T_2$  and is not a surjection if  $A(G) = T_3$ .

Recall that a map  $\phi : A_1 \rightarrow A_2$  between digraph algebras is said to be *regular* if there exist masas  $C_i \subseteq A_i$  such that  $\phi$  maps the partial isometry normaliser  $N_{C_1}(A_1)$  into the partial isometry normaliser  $N_{C_2}(A_2)$  [40], [39]. The regular maps between digraph algebras are precisely those which are 1-decomposable. Write  $\mathcal{F}_G^{\text{reg}}$  for the family of regular maps  $A(G) \otimes M_n \rightarrow A(G) \otimes M_m$ , for all  $m, n$ . Then the metrized embedding semiring for this family of maps is the semiring  $\mathbb{Z}_+[\text{End } G]$ , with the discrete metric. This family is closed in the sense of Section 2, as well as being topologically closed.

**Example 3.8 Toeplitz algebras.** Let  $L_2$  be the finitely acting operator algebra in  $M_2$  consisting of the matrices

$$\begin{bmatrix} a & b \\ 0 & a \end{bmatrix}.$$



The embedding semiring  $V_{L_2}$  contains the classes of embeddings

$$\rho_\theta \otimes i : E \otimes M_n \rightarrow E \otimes M_m$$

where  $n < m$ ,  $i : M_n \rightarrow M_m$  is a multiplicity one inclusion and

$$\rho_\theta : \begin{bmatrix} a & b \\ 0 & a \end{bmatrix} \rightarrow \begin{bmatrix} a & \theta b \\ 0 & a \end{bmatrix}$$

for  $|\theta| = 1$ . This subset of classes, with the relative topology, is seen to be a homeomorph of the unit circle  $S^1$ . The embeddings  $\rho_\theta$  are indecomposable and so  $d(L_2) = \infty$ . Moreover it can be verified that each indecomposable map of multiplicity one is equivalent to  $\rho_\theta$  for some  $\theta$ . The only other indecomposable is equivalent to the multiplicity two map  $\tau : E \rightarrow E \otimes M_2$  with range in  $\mathbb{C} \otimes M_2$ . It follows that  $V_{L_2}$  is the abelian semiring

$$V_{L_2} = \mathbb{Z}_+[S^1] \oplus \mathbb{Z}_+[[\tau]]$$

with product such that  $[\rho_\theta][\tau] = [\tau][\rho_\theta] = [\tau]$ .

**Example 3.9 Quiver algebras.** Finally let us indicate how certain finitely acting operator algebras are derived from quivers.

Let  $Q = (V, E)$  be a quiver, that is an arbitrary finite directed graph with vertex set  $V$  and edge set  $E$ . A (finite directed) path  $p$  of  $Q$  is either a trivial path  $1_v$ , with initial vertex and final vertex  $v$ , or is a sequence  $e_t \dots e_1$  of edges  $e$  of  $E$ , for which the final vertex of  $e_i$  is equal to the initial vertex of  $e_{i+1}$ , for  $1 \leq i \leq t-1$ . The path algebra  $\mathbb{C}Q$  is defined to be the complex algebra of formal linear combinations of paths with the product defined by cocatenation of paths. This is a finite-dimensional algebra precisely when  $Q$  is acyclic.

The finite-dimensional algebras  $A = \mathbb{C}Q$  with  $(\text{rad } A)^2 = 0$  correspond to quivers which are bipartite directed graphs. These include the poset algebras  $A(\mathcal{P})$  with this property, since in this case  $A(\mathcal{P})$  is isomorphic to  $\mathbb{C}Q$  where  $Q$  is the directed graph determined by  $\mathcal{P}$  with loop edges at vertices removed. That the path algebras here are more general is due to the possibility of multiple edges. A basic example here is the Kronecker algebra, which is the complex quiver path algebra for the quiver with two edges incident on a single

vertex and which may be realised as the subalgebra of  $M_3$  consisting of the matrices

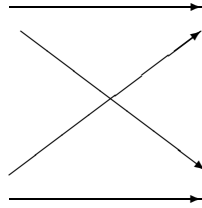
$$\begin{bmatrix} a & x & y \\ 0 & b & 0 \\ 0 & 0 & b \end{bmatrix}.$$

There is a more general association of finite dimensional complex algebras with quivers which is given in terms of quotients of the path algebras of general quivers. Those with  $(\text{rad } A)^2 = 0$  (which have importance in the representation theory of quivers) we may define directly as the complex algebras  $A_Q$  which are representable in terms of a matrix algebra as the set of matrices of the form

$$\sum_{e=(u,v) \in E} \oplus \begin{bmatrix} \lambda_u & \lambda_e \\ 0 & \lambda_v \end{bmatrix}$$

where  $\lambda_u, \lambda_e, \lambda_v \in \mathbb{C}$ ,  $Q$  is a connected quiver and the direct sum is taken over all directed edges  $e = (u, v)$  of the quiver. These algebras contain the finite-dimensional path algebras with  $(\text{rad } A)^2 = 0$  indicated above. That they are more general is due to the admission of non-acyclic quivers. In particular the  $A_Q$  algebra for the loop quiver with a single vertex and edge is the elementary Toeplitz algebra  $L_2$ .

The presentation above gives outright a particular representation of  $A_Q$  as a finitely acting operator algebra. Let us denote this operator algebra as  $A_Q^{\text{min}}$ . In particular if  $Q$  is the bipartite quiver with four edges and 4 vertices,



then  $A_Q^{\text{min}}$  is the operator subalgebra of  $\mathbb{C}^4 \otimes M_2$  consisting of the matrices

$$\begin{bmatrix} a & x \\ 0 & c \end{bmatrix} \oplus \begin{bmatrix} a & y \\ 0 & d \end{bmatrix} \oplus \begin{bmatrix} b & z \\ 0 & d \end{bmatrix} \oplus \begin{bmatrix} b & w \\ 0 & c \end{bmatrix}.$$

This algebra can be viewed as an inflation algebra (in the sense of Definition 11.1) of the 4-cycle digraph algebra  $A(D_4)$ . One can readily verify that  $A_Q^{\text{min}}$  has infinite embedding

type, in the tame sense of Example 3.8, whereas the maximal inflation algebra  $A(D_4)^{max}$ , like  $T_4^{max}$  has finite embedding type.

#### 4. COMPLETE INVARIANTS FOR LOCALLY FINITE ALGEBRAS

We now obtain complete invariants for algebraic direct limit algebras whose building blocks are  $E$ -algebras. The main algebraic ideas concerning the existence and uniqueness steps will reappear in the consideration of operator algebra limits in the next section.

Let  $E$  be a finitely acting operator algebra and let  $\mathcal{F}$  be a family of star extendible embeddings  $\phi$  between finite dimensional algebras of the form  $E \otimes M_n$  and assume that  $\mathcal{F}$  satisfies the metric and algebraic closure properties (i)-(v) of Section 2. We do not assume that  $E$  is of finite embedding rank since this confers no particular simplification here.

Let  $A \in \text{Lim } \tilde{\mathcal{F}}$  so that  $A$  has a presentation

$$A = \varinjlim (A_k, \phi_k)$$

where each  $A_k$ , for  $k = 1, 2, \dots$ , is an elementary  $E$ -algebra (that is, a finite direct sum of, let us say,  $r_k$  matrix algebras over  $E$ ) and where the partial embeddings of each map  $\phi_k$  belong to  $\mathcal{F}$ . Also let

$$A_0 = \text{alg } \varinjlim (A_k, \phi_k)$$

be the associated locally finite algebra.

Define  $V_{\mathcal{F}}(A_k)$  to be the right  $V_{\mathcal{F}}$ -module which is the direct sum of  $r_k$  copies of  $V_{\mathcal{F}}$  and consider  $V_{\mathcal{F}}(A_k)$  (more functorially) as the monoid of inner unitary equivalence classes of star extendible embeddings  $\psi : E \rightarrow A_k \otimes \mathcal{K}$ , where the partial embeddings belong to  $\mathcal{F}$ . Define the scale  $\Sigma_{\mathcal{F}}(A_k)$  of  $V_{\mathcal{F}}(A_k)$  as the subset of classes  $[\psi]$  for which  $\psi : E \rightarrow A_k$  where  $A_k$  is identified with  $A_k \otimes \mathbb{C}p$  for some rank one projection  $p$ .

For each  $k$  we have the induced  $V_{\mathcal{F}}$ -module homomorphism

$$\hat{\phi}_k : V_{\mathcal{F}}(A_k) \rightarrow V_{\mathcal{F}}(A_{k+1})$$

given by  $\hat{\phi}_k([\psi]) = [\phi_k \circ \psi]$ . Plainly  $\hat{\phi}$  respects the right  $V_{\mathcal{F}}$ -action, which is to say that for  $[\theta]$  in  $V_{\mathcal{F}}$ ,

$$\hat{\phi}_k([\psi][\theta]) = (\hat{\phi}_k([\psi]))[\theta].$$

**Definition 4.1.** The dimension module of the direct system  $\{A_k, \phi_k\}$ , for the family  $\mathcal{F}$ , is the right  $V_{\mathcal{F}}$ -module

$$V_{\mathcal{F}}(\{A_k, \phi_k\}) = \varinjlim (V_{\mathcal{F}}(A_k), \hat{\phi}_k).$$

The direct limit here is taken in the category of additive abelian semigroups and endowed with the induced right  $V_{\mathcal{F}}$ -action.

Define the scale  $\Sigma_{\mathcal{F}}(A_0)$  of  $V_{\mathcal{F}}(A_0)$  to be the union of the images of the scales  $\Sigma_{\mathcal{F}}(A_k)$  in  $V_{\mathcal{F}}(A_0)$ . Writing  $G_{\mathcal{F}}(A_0)$  for the enveloping group of  $V_{\mathcal{F}}(A_0)$  we obtain the scaled ordered group

$$(G_{\mathcal{F}}(A_0), V_{\mathcal{F}}(A_0), \Sigma_{\mathcal{F}}(A_0))$$

together with the right  $V_{\mathcal{F}}$ -action on  $V_{\mathcal{F}}(A_0)$  as a tentative invariant for star-extendible isomorphism. In view of Proposition 3.4 the additive semigroup  $V_{\mathcal{F}}(A_0)$  has cancellation and so the inclusion  $V_{\mathcal{F}}(A_0) \rightarrow G_{\mathcal{F}}(A_0)$  is injective. In fact we shall focus particularly on the possibility that the dimension module  $V_{\mathcal{F}}(A_0)$  is invariant for star-extendible homomorphisms.

The reason for caution here is that we have no reason yet to expect functoriality in the sense that a star extendible homomorphism  $\Phi : A_0 \rightarrow A'_0$  naturally induces an (additive group) homomorphism from  $V_{\mathcal{F}}(A_0)$  to  $V_{\mathcal{F}}(A'_0)$ . Indeed we see in the Section 11 that a commuting diagram between the two direct systems which is induced by  $\Phi$  may involve morphisms which are not associated with  $\mathcal{F}$ . One way around this is to require a further property for  $\mathcal{F}$ , namely the functoriality property of Definition 9.1. With a property such as this it becomes clear that  $V_{\mathcal{F}}(A_0)$  is indeed an invariant for the algebra, as the notation suggests, and is not dependent in an essential way on the particular direct system for  $A_0$ .

The next theorem shows that the scaled dimension module is a complete invariant for algebraic limit algebras determined by a functorial family. In the proof we have versions of the familiar existence and uniqueness steps in the construction of a commuting diagram between direct systems. The existence step is relatively novel in that it relies on the fact that the isomorphism respects the  $V_{\mathcal{F}}$ -action. The uniqueness step is quite elementary and closely analogous to the self-adjoint case.

**Proposition 4.2. (Uniqueness.)** *Let  $E$  be a finitely acting operator algebra and let  $\phi, \psi : A_1 \rightarrow A_2$  be maps between elementary  $E$ -algebras with induced maps  $\hat{\phi}, \hat{\psi}$  from  $V_{\mathcal{F}}(A_1)$  to  $V_{\mathcal{F}}(A_2)$ . Then  $\hat{\phi} = \hat{\psi}$  if and only if  $\phi, \psi$  are inner unitarily equivalent.*

*Proof.* It will be enough to establish the proposition when  $A_1 = E \otimes M_{n_1}$  and  $A_2 = E \otimes M_{n_2}$ . Let  $\theta : E \otimes \mathcal{K} \rightarrow E \otimes \mathcal{K}$  be the identity embedding. Then  $\hat{\phi}([\theta]) = \hat{\psi}([\theta])$  and so  $[\phi \circ \theta] = [\psi \circ \theta]$  which is to say that  $[\phi] = [\psi]$  and hence that the induced maps  $\phi', \psi' : A_1 \otimes \mathcal{K} \rightarrow A_2 \otimes \mathcal{K}$  are unitarily equivalent. From this it follows that  $\phi, \psi$  are unitarily equivalent.  $\square$

**Theorem 4.3.** *Let  $\mathcal{F}$  be a family of star-extendible embeddings between operator algebras  $E \otimes M_n$ , for  $n = 1, 2, \dots$ , which is closed in the sense of Section 2, and let  $A_0, A'_0$  belong to  $\text{Alglim}(\tilde{\mathcal{F}})$ . If  $\Gamma$  is a  $V_{\mathcal{F}}$ -module isomorphism from  $V_{\mathcal{F}}(A_0)$  to  $V_{\mathcal{F}}(A'_0)$  then  $A_0 \otimes \mathcal{K}$  and  $A'_0 \otimes \mathcal{K}$  are star-extendibly isomorphic. If, moreover,  $\Gamma$  gives a bijection from  $\Sigma_{\mathcal{F}}(A_0)$  to  $\Sigma_{\mathcal{F}}(A'_0)$  then  $A_0$  and  $A'_0$  are star-extendibly isomorphic.*

*If  $\mathcal{F}$  is functorial then the converse of these assertions hold and the  $V_{\mathcal{F}}$ -module  $V_{\mathcal{F}}(-)$  is a complete invariant for stable star-extendible isomorphism, whilst the pair  $(V_{\mathcal{F}}(-), \Sigma_{\mathcal{F}}(-))$  is a complete invariant for star-extendible isomorphism.*

*Proof.* Let  $A_0 = \varinjlim (A_k, \phi_k)$ ,  $A'_0 = \varinjlim (A'_k, \phi'_k)$  be the given presentations. Consider the  $V_{\mathcal{F}}$ -module homomorphism  $\gamma$  which is the composition

$$\begin{array}{ccc} V_{\mathcal{F}}(A_1) & \longrightarrow & V_{\mathcal{F}}(A_0) \\ & \searrow \gamma & \downarrow \Gamma \\ & & V_{\mathcal{F}}(A'_0) \end{array}$$

where  $V_{\mathcal{F}}(A_1) \rightarrow V_{\mathcal{F}}(A_0)$  is the natural map. Suppose first that  $A_1 = E \otimes M_{n_1}$  and let  $\theta : E \rightarrow A_1$  be the map  $a \rightarrow a \otimes p$  where  $p$  is a rank one projection. Then  $[\theta]$ , the class of  $\theta$  in  $V_{\mathcal{F}}(A_1)$ , has image  $\gamma([\theta])$  in  $V_{\mathcal{F}}(A'_0)$  which in turn coincides with the image of a class  $[\eta_1]$  from  $V_{\mathcal{F}}(A'_k)$ , for some  $k$ , under the natural map  $V_{\mathcal{F}}(A'_k) \rightarrow V_{\mathcal{F}}(A'_0)$ . The representative  $\eta_1$  of  $[\eta_1]$  is a map  $\eta_1 : E \rightarrow A'_k \otimes M_{n_1}$ , for some  $n_1$ , which in turn gives an induced map  $\eta_2$  from  $E \otimes M_n$  to  $A'_k \otimes M_{n_1 n}$ .

Suppose first that  $A_0$  and  $A'_0$  are stable algebras, so that  $A_0 = A_0 \otimes \mathcal{K}$ ,  $A'_0 = A'_0 \otimes \mathcal{K}$ , and  $\Sigma_{\mathcal{F}}(A_0) = V_{\mathcal{F}}(A_0)$ ,  $\Sigma_{\mathcal{F}}(A'_0) = V_{\mathcal{F}}(A'_0)$ . In particular this means that for any positive integers  $k$  and  $N$  one can find  $l > k$  so that if  $\alpha : A_k \rightarrow A_l$  is the given embedding then the induced map  $\alpha \otimes \text{id} : A_k \otimes M_N \rightarrow A_l \otimes M_N$  is inner equivalent to a map  $\gamma : A_k \otimes M_N \rightarrow A_l$ .

Increasing  $k$  if necessary it follows that we can replace  $\eta_2$  by an inner equivalent map  $\eta : E \otimes M_n \rightarrow A'_k$  such that the associated class  $[\eta]$  in  $V_{\mathcal{F}}(A'_k)$  has image  $\gamma([\eta])$  in  $V_{\mathcal{F}}(A'_0)$ . It now follows that we have the factorisation

$$\begin{array}{ccc} V_{\mathcal{F}}(A_1) & & \\ \downarrow \hat{\eta} \quad \searrow \gamma & & \\ V_{\mathcal{F}}(A'_k) & \longrightarrow & V_{\mathcal{F}}(A'_0) \end{array}$$

Indeed, since  $\gamma$  and  $\hat{\eta}$  are  $V_{\mathcal{F}}$ -module maps we have, for  $[\psi]$  in  $V_{\mathcal{F}}$ ,

$$\gamma([\psi]) = \gamma([\theta][\psi]) = (\gamma([\theta]))[\psi],$$

whilst the image in  $V_{\mathcal{F}}(A'_0)$  of  $\hat{\eta}([\psi])$  is the image of

$$\hat{\eta}([\theta][\psi]) = \hat{\eta}([\theta])[\psi] = [\eta][\psi],$$

and by construction,  $[\eta]$  has image  $\gamma([\theta])$ . In other words, the map  $\eta$  is a lifting for  $\gamma$ .

The case when  $A_1$  has more than one summand now follows by combining the liftings of partial embeddings.

Repeat the argument above for the  $V_{\mathcal{F}}$ -module homomorphism  $\delta$  which is the composition

$$\begin{array}{ccc} & V_{\mathcal{F}}(A) & \\ \nearrow \delta & \uparrow \Gamma^{-1} & \\ V_{\mathcal{F}}(A'_k) & \longrightarrow & V_{\mathcal{F}}(A'_0) \end{array}$$

where  $V_{\mathcal{F}}(A'_k) \rightarrow V_{\mathcal{F}}(A'_0)$  is the natural map, to obtain a map  $\kappa : A'_k \rightarrow A_j$  which is a lifting of  $\delta$ .

Since  $\hat{\kappa} \circ \hat{\eta}$  is equal to the given map from  $V_{\mathcal{F}}(A_1) \rightarrow V_{\mathcal{F}}(A_j)$  it follows from Proposition 4.2 that we can replace  $\delta$  by a unitarily equivalent map to obtain a commuting triangle. Since the process can be repeated we obtain an infinite commuting diagram of maps between the two given direct systems from which it follows that  $A_0$  and  $A'_0$  are star-extendibly isomorphic.

Suppose now that  $A_0$  and  $A'_0$  are not necessarily stable and that  $\Gamma$  preserves the scales. Once again consider first the single summand case  $A_1 = E \otimes M_{n_1}$ . If  $\theta$  is as above note that  $n_1[\theta]$  lies in  $\Sigma_{\mathcal{F}}(A_1)$  and so we may choose  $k$  large enough so that  $n_1[\eta]$  is in  $\Sigma_{\mathcal{F}}(A'_{k_1})$ .

It follows that there is an extension  $\tilde{\eta} : A_1 \rightarrow A'_{k_1}$  and the proof may be completed as before.  $\square$

## 5. METRIZED DIMENSION MODULE INVARIANTS

The dimension module classification theorem for algebraic direct limits also serves to provide sufficient conditions for the star extendible isomorphism of operator algebra limits. In general these conditions are not necessary conditions since the closures of  $A_0$  and  $A'_0$  may be isomorphic when  $A_0$  and  $A'_0$  are not (as we noted in Example 3.6). In this section we obtain the appropriate invariants by replacing  $V_{\mathcal{F}}(A_0)$  by its completion under a pseudometric induced by the metric structures on  $V_{\mathcal{F}}(A_k)$ . On the other hand, if  $E$  is of finite embedding type and perturbationally stable then we shall see in the next section that this step is unnecessary and the functor  $G : \text{Alglim}(\mathcal{F}_E) \rightarrow \text{Lim}(\mathcal{F}_E)$  is injective.

Let  $E, \mathcal{F}, A = \varinjlim (A_k, \phi_k)$  be as given at the beginning of the last section.. Provide the  $V_{\mathcal{F}}$ -modules  $V_{\mathcal{F}}(A_k)$  with the natural metrics  $d_k$  given by the formula of Definition 3.1, although now  $V_{\mathcal{F}}(A_k)$  may be a finite direct sum of copies of  $V_{\mathcal{F}}$ . We now define the metrized dimension module  $(V_{\mathcal{F}}(A), d)$  of the algebra  $A$  together with its presentation.

View  $\hat{\phi}_k$  as a contractive metric space map from  $(V_{\mathcal{F}}(A_k), d_k)$  to  $(V_{\mathcal{F}}(A_{k+1}), d_{k+1})$  and define the complete metrized monoid

$$(V_{\mathcal{F}}(A), d) = \varinjlim ((V_{\mathcal{F}}(A_k), d_k), \hat{\phi}_k)$$

with the limit taken in the category of metrically complete abelian semigroups. Explicitly this means that one forms the abelian semigroup direct limit

$$V_{\mathcal{F}}^{\infty}(A) = \varinjlim (V_{\mathcal{F}}(A_k), \hat{\phi}_k)$$

(which is the same as  $V_{\mathcal{F}}(\{A_k, \phi_k\})$ ) together with the pseudometric  $d$  for which

$$d(\hat{\phi}_{k,\infty}([\psi]), \hat{\phi}_{k,\infty}([\eta])) = \lim_{l \rightarrow \infty} d_l(\hat{\phi}_{k,l}([\psi]), \hat{\phi}_{k,l}([\eta])).$$

Here  $\hat{\phi}_{k,\infty} : V_{\mathcal{F}}(A_k) \rightarrow V_{\mathcal{F}}^{\infty}(A)$  and  $\hat{\phi}_{k,l} : V_{\mathcal{F}}(A_k) \rightarrow V_{\mathcal{F}}(A_l)$  are the natural homomorphisms. Let us relax notation and write  $[\eta]$  for the typical element  $\hat{\phi}_{k,\infty}([\eta])$  of  $V_{\mathcal{F}}^{\infty}(A)$ . Define the equivalence relation  $[\phi] \sim [\psi]$  as that for which  $d([\phi], [\psi]) = 0$ . It follows that the equivalence classes inherit a well-defined abelian semigroup structure. In this way we

obtain the abelian semigroup  $V_{\mathcal{F}}^{\infty}(A)/\sim$ , with induced metric  $d$ . The completion of this metric space gives the metrized semigroup  $(V_{\mathcal{F}}(A), d)$  which carries a unique continuous right action by  $V_{\mathcal{F}}$  which is induced from the  $V_{\mathcal{F}}$ -action on  $V_{\mathcal{F}}^{\infty}$ . The scale  $\Sigma_{\mathcal{F}}(A)$  in  $V_{\mathcal{F}}(A)$  is defined naturally as the closure in  $V_{\mathcal{F}}(A)$  of the natural scale  $\Sigma_{\mathcal{F}}^{\infty}(A)/\sim$  in  $V_{\mathcal{F}}^{\infty}$ .

**Theorem 5.1.** *Let  $E$  be a finitely acting operator algebra and let  $\mathcal{F}$  be a family of star-extendible embeddings between operator algebras  $E \otimes M_n$ , for  $n = 1, 2, \dots$ , which is closed in the sense of Section 2. Let  $A, A'$  belong to  $\text{Lim}(\tilde{\mathcal{F}})$  with direct systems determining the complete metrized semirings  $(V_{\mathcal{F}}(A), d)$  and  $(V_{\mathcal{F}}(A'), d')$ . If  $\Gamma$  is a  $V_{\mathcal{F}}$ -module isomorphism from  $V_{\mathcal{F}}(A)$  to  $V_{\mathcal{F}}(A')$  which is a bicontinuous metric space map then  $A \otimes \mathcal{K}$  and  $A' \otimes \mathcal{K}$  are star-extendibly isomorphic. If, moreover,  $\Gamma$  gives a bijection from  $\Sigma_{\mathcal{F}}(A)$  to  $\Sigma_{\mathcal{F}}(A')$  then  $A$  and  $A'$  are star-extendibly isomorphic.*

*Proof.* Let  $A = \varinjlim (A_k, \phi_k)$ ,  $A' = \varinjlim (A'_k, \phi'_k)$  be the given presentations. Consider the  $V_{\mathcal{F}}$ -module homomorphism  $\gamma$  which is the composition

$$\begin{array}{ccc} V_{\mathcal{F}}(A_1) & \longrightarrow & V_{\mathcal{F}}(A) \\ & \searrow \gamma & \downarrow \Gamma \\ & & V_{\mathcal{F}}(A') \end{array}$$

where  $V_{\mathcal{F}}(A_1) \rightarrow V_{\mathcal{F}}(A)$  is the natural map. Suppose first that  $A_1 = E \otimes M_{n_1}$  and let  $\theta : E \rightarrow A_1$ , be the map  $a \rightarrow a \otimes p$  where  $p$  is a rank one projection. Then  $[\theta]$ , the class of  $\theta$  in  $V_{\mathcal{F}}(A_1)$ , has image  $\gamma([\theta])$  in  $V_{\mathcal{F}}(A'_0)$ . Let  $\epsilon_1 > 0$ . Choose  $[\eta]$  in  $V_{\mathcal{F}}(A'_{k_1})$ , for large enough  $k_1$ , so that

$$d'(\hat{\phi}'_{k_1, \infty}([\eta]), \gamma_1([\theta])) < \epsilon_1.$$

View  $\eta : E \rightarrow A'_{k_1}$  as a partially defined map from  $E \otimes M_n$  to  $A'_{k_1}$ , defined on  $E \otimes \mathbb{C}p$ . Consider first the case of stable algebras. Then we may increase  $k_1$  if necessary to obtain a unique extension  $\tilde{\eta}$  of  $\eta$ , with  $\tilde{\eta} : A_1 \rightarrow A'_{k_1}$ . In particular we have

$$\hat{\tilde{\eta}}([\theta]) = [\eta],$$

as classes in  $V_{\mathcal{F}}(A'_{k_1})$  and  $\hat{\phi}'_{k_1, \infty}([\eta]) = \hat{\phi}'_{k_1, \infty}([\tilde{\eta}])$ . By  $V_{\mathcal{F}}$ -action preservation, for  $[\psi]$  in  $V_{\mathcal{F}}$ ,

$$\gamma_1([\psi]) = \gamma_1([\theta][\psi]) = \gamma_1([\theta])[\psi]$$



which is  $\epsilon_1$ -close to

$$[\eta][\psi] = [\tilde{\eta}][\psi] = \hat{\tilde{\eta}}([\psi]),$$

where we identify  $[\eta], [\tilde{\eta}]$  with their image classes in  $V_{\mathcal{F}}(A')$ . This is true for all  $\psi$  in  $V_{\mathcal{F}}$  and so  $\tilde{\eta}$  is an approximate lifting of  $\gamma_1$  in the sense that the maps  $\gamma_1$  and  $\hat{\phi}'_{k_1, \infty} \circ \hat{\tilde{\eta}}$  are  $\epsilon_1$ -close as metric space maps from  $V_{\mathcal{F}}(A_1)$  to  $V_{\mathcal{F}}(A')$ . The case when  $A_1$  has more than one summand now follows by combining the liftings of partial embeddings.

Repeat the argument above for  $\epsilon_2$  and the map  $\delta_1 = \Gamma^{-1} \circ \hat{\phi}'_{k, \infty}$  from  $V_{\mathcal{F}}(A'_{k_1})$  to  $V_{\mathcal{F}}(A)$  to obtain an  $\epsilon_2$ -approximate lifting  $\tilde{\xi}$  from  $A'_{k_1}$  to  $A_{k_2}$ .

Now

$$d([\tilde{\xi} \circ \tilde{\eta}], [\phi_{1, k_2}]) \leq \epsilon_1 + \epsilon_2$$

and so, increasing  $k_2$  we may obtain

$$d_{k_2}([\tilde{\xi} \circ \tilde{\eta}], [\phi_{1, k_2}]) \leq \epsilon_1 + 2\epsilon_2,$$

and so we may choose a representative  $\kappa$  in  $[\tilde{\xi}]$  so that  $\|\kappa \circ \tilde{\eta} - \phi_{1, k_2}\| \leq \epsilon_1 + 3\epsilon_2$ .

Continue in this way, for a suitable sequence  $\epsilon_k$ , to obtain an approximately commuting diagram and the desired isomorphism.

Suppose now that  $A, A'$  are not necessarily stable. Once again consider first the single summand case  $A_1 = E \otimes M_{n_1}$  and  $\epsilon_1 > 0$ . Noting that  $n_1[\theta]$  lies in  $\Sigma_{\mathcal{F}}(A_1)$  choose  $k_1$  large enough so that  $[\eta]$  and  $n_1[\eta]$  are in  $\Sigma_{\mathcal{F}}(A'_{k_1})$  and

$$d'(\hat{\phi}'_{k_1, \infty}([\eta]), \gamma_1([\theta])) < \epsilon_1.$$

Since  $n_1[\eta]$  lies in the scale there is an extension  $\tilde{\eta}$  and the proof may be completed as before.  $\square$

**Corollary 5.2.** *Let  $E$  be a finitely acting operator algebra and let  $A, A'$  be operator algebra direct limits of  $E$ -algebras with respect to star extendible embeddings. If the scaled metrized dimension  $V_E$ -modules of  $A$  and  $A'$  are bicontinuously isomorphic then  $A, A'$  are star extendibly isomorphic.*

We note that in the case of unrestricted embeddings the invariants seem to have a complexity comparable to the limit algebras themselves. However even the general classification has theoretical strength as we see in the next section where we obtain perturbational stability for various limit algebras. On the other hand, with prescribed embedding classes, or with prescribed building block algebras, the  $V_{\mathcal{F}}$ -module action may be replaced by a semigroup action or group action and the invariants become computable, as we shall see.

## 6. STABILITY AND COMPLETE INVARIANTS

Let  $\mathcal{E}$  be a family of finitely acting operator algebras.

**Definition 6.1.** The family  $\mathcal{E}$  has the stability property, or is perturbationally stable, if for each algebra  $A_1$  in  $\mathcal{E}$  and  $\epsilon > 0$  there is a  $\delta > 0$  such that to each algebra  $A_2$  in  $\mathcal{E}$  and star-extendible embedding

$$\phi : A_1 \rightarrow C^*(A_2), \quad \text{with} \quad \phi(A_1) \subseteq_{\delta} A_2$$

there is a star-extendible embedding  $\psi : A_1 \rightarrow A_2$  with  $\|\phi - \psi\| \leq \epsilon$ .

In addition we say that the family is *uniformly stable* if  $\delta$  may be chosen independently of the algebras in the family, and we say that a finite-dimensional operator algebra  $E$  is perturbationally stable if the family of elementary  $E$ -algebras is perturbationally stable. For such an algebra  $E$  it can be shown that  $AF_E = \text{Lim } \mathcal{F}_E$ .

The operator algebra  $\mathbb{C}$  is well known to be stable and we shall see further examples later. For a stable algebra  $E$  one has the following natural identifications for an operator algebra inductive  $A = \varinjlim A_k$  of elementary  $E$  algebras:

$$Hom_a(E, A \otimes \mathcal{K}) = \varinjlim Hom_u(E, A_k \otimes \mathcal{K}), \quad Hom_a(E, A) = \varinjlim Hom_u(E, A_k).$$

Here  $Hom_u(-, -)$  indicates the inner unitary equivalence classes of star-extendible homomorphisms and  $Hom_a(-, -)$  the equivalence classes of star-extendible homomorphisms with respect to approximately inner automorphisms.

We can now obtain the following generalisation of Elliott's classification of AF  $C^*$ -algebras.

**Theorem 6.2.** *Let  $E$  be a perturbationally stable finitely acting operator algebra and let  $A$  belong to  $AF_E$ . Then the metrized dimension module  $(V_E(A), d)$  is isometrically isomorphic*

to the metrized  $V_E$ -module  $\text{Hom}_a(E, A \otimes \mathcal{K})$ . Furthermore, two algebras  $A, A'$  in  $AF_E$  are star-extendibly isomorphic if and only if there is a bicontinuous semigroup isomorphism

$$\Gamma : \text{Hom}_a(E, A \otimes \mathcal{K}) \rightarrow \text{Hom}_a(E, A' \otimes \mathcal{K}),$$

with

$$\Gamma(\text{Hom}_a(E, A)) = \text{Hom}_a(E, A'),$$

which respects the right action of  $\text{Hom}_a(E, E \otimes \mathcal{K})$ .

*Proof.* It is clear that if  $A$  and  $A'$  are star-extendibly isomorphic then there is an induced isomorphism  $\Gamma$  of the invariants. The sufficiency direction will follow from Corollary 5.2 once we show that  $V_E(A)$  is naturally isomorphic to  $\text{Hom}_a(E, A \otimes \mathcal{K})$  with corresponding identification of scales.

Let  $\Phi_0 : V_E^\infty(A) \rightarrow \text{Hom}_a(E, A \otimes \mathcal{K})$  be the morphism for which  $\Phi_0([\phi]) = [\phi]_{\text{Hom}}$ , where  $\phi : E \rightarrow A_k \otimes M_n$  and  $[\phi]$  denotes the image in  $V_E^\infty(A)$  of the class  $[\phi]_{A_k}$  of  $\phi$  in  $V_E(A_k)$ , and where  $[\phi]_{\text{Hom}}$  denotes the class of  $i_k \circ \phi$  where  $i_k : A_k \otimes M_n \rightarrow A \otimes \mathcal{K}$  is the natural injection. If  $[\phi] = [\psi]$ , then there is a sequence  $u_n$  of unitaries in  $A_{k+n}$  for which

$$d([\phi]_{A_{k+n}}, [u_n \psi u_n^*]_{A_{k+n}}) \rightarrow 0$$

as  $n \rightarrow \infty$ , which is to say that the morphisms  $i_k \circ \phi, i_k \circ \psi$  from  $E$  to  $A \otimes M_n$  are approximately unitarily equivalent and so  $[\phi]_{\text{Hom}} = [\psi]_{\text{Hom}}$ . For the same reasons we see that the metric  $d(-, -)$  on  $V_E^\infty(A)$  coincides under  $\Phi_0$  with the metric on  $\text{Hom}_a(E, A \otimes \mathcal{K})$ .

Extending  $\Phi_0$  by continuity we obtain a continuous isometric injection

$$V_E(A) \rightarrow \text{Hom}_a(E, A \otimes \mathcal{K}).$$

In view of the fact that  $E$  is stable this map is also surjective and thus bijective. It also follows from the stability of  $E$  that the scale of  $V_E(A)$  corresponds to  $\text{Hom}_a(E, A)$ .  $\square$

It has been shown by Haworth [21] that the finitely acting operator algebra  $T_r$  (in  $M_r$ ) is perturbationally stable and hence that  $\text{Lim } \mathcal{F}_{T_r} = AF_{T_r}$  for each  $r \geq 1$ . Combining this with the last theorem we obtain Theorem 6.3. The theorem reduces to Elliott's theorem in the case  $r = 1$ . For  $r = 2$  star-extendible embeddings are automatically regular (as we have seen in Example 3.5) and  $\text{Hom}_u(T_2, T_2 \otimes \mathcal{K})$  identifies with  $\mathbb{Z}_+^3$  with discrete metric.

This classification provides an alternative to the algebraically ordered scaled ordered  $K_0$  group of Power [40] and Heffernan [22].

We refer to operator algebras in  $\text{Lim } \mathcal{F}_{T_r} = \text{AF}_{T_r}$  as approximately finite nest algebras of diameter  $r$ .

**Theorem 6.3.** *Let  $A, A'$  be approximately finite nest algebras of diameter  $r$ . Then  $A, A'$  are star-extendibly isomorphic if and only if there is a bicontinuous semigroup isomorphism*

$$\Gamma : \text{Hom}_a(T_r, A \otimes \mathcal{K}) \rightarrow \text{Hom}_a(T_r, A' \otimes \mathcal{K}),$$

with

$$\Gamma(\text{Hom}_a(T_r, A)) = \text{Hom}_a(T_r, A'),$$

which respects the right action of  $\text{Hom}_a(T_r, T_r \otimes \mathcal{K})$ .

More generally the algebras of  $\text{Lim } \tilde{\mathcal{F}}_{T_r}$  are classified in the same way.

We now show that limit algebras determined by uniformly stable families are themselves stable in the sense that if  $A \subseteq_\delta A'$  and  $A' \subseteq_\delta A$  then  $A$  and  $A'$  are star extendibly isomorphic. The hypothesis here is that the algebras are star extendibly included in a common  $C^*$ -algebra and that the Hausdorff distance between their unit balls is no greater than  $\delta$ . The key idea of the proof is to use the local compactness of  $V_{\mathcal{F}}(A)$  and  $V_{\mathcal{F}}(A')$  together with the Ascoli-Arzelà theorem to construct an isomorphism between the invariants, and to lift this to the desired algebra isomorphism.

**Theorem 6.4.** *Let  $E$  be a perturbationally stable finitely acting operator algebra and let  $\mathcal{F} = \mathcal{F}_E$ . Let  $A, A'$  be operator algebras in the class  $\text{Lim } \tilde{\mathcal{F}}$  acting on a common Hilbert space and suppose that  $A \subseteq_\delta A'$ ,  $A' \subseteq_\delta A$  for some  $\delta > 0$ . If  $\delta$  is sufficiently small then  $A$  and  $A'$  are star extendibly isomorphic.*

*Proof.* Let  $A = \varinjlim (A_k, \phi_k)$ ,  $A' = \varinjlim (A'_k, \phi'_k)$  be presentations with partial embeddings in  $\mathcal{F}$ . For each  $k$  choose  $n_k$  large enough so that

$$A_k \subseteq_{2\delta_1} A'_{n_k}.$$

By uniform stability we assume that  $\delta_1$  is chosen so that we may obtain in  $\mathcal{F}$  a star-extendible embedding  $\psi_k : A_k \rightarrow A'_{n_k}$  with  $\|\psi_k - \text{id}\| \leq 1/3$ . The map  $\psi_k$  induces a map  $\hat{\psi}_k$ ,

$$\hat{\psi}_k : V_{\mathcal{F}}(A_k) \rightarrow V_{\mathcal{F}}(A'_{n_k}).$$

Furthermore, this map is graded by multiplicity in the sense that if  $s$  is the multiplicity of  $\psi_k$  then for each  $r$

$$\hat{\psi}_k^{(r)} : V_{\mathcal{F}}(A_k)^{(r)} \rightarrow V_{\mathcal{F}}(A'_{n_k})^{(r+s)}.$$

The maps  $\hat{\psi}_k$  are equicontinuous metric space maps. For suppose that  $[\eta], [\nu] \in V_{\mathcal{F}}(A_k)^{(r)}$ . Then

$$\begin{aligned} d'_{n_k}(\hat{\psi}_k([\eta]), \hat{\psi}_k([\nu])) &= \inf_{u \in U(A'_{n_k})} \|\psi_k \circ \eta - (Adu) \circ \psi_k \circ \nu\| \\ &\leq \inf_{v \in U(A_k)} \|\psi_k \circ (\eta - (Adv) \circ \nu)\| \\ &\leq \inf_{v \in U(A_k)} \|\eta - (Adv) \circ \nu\| \\ &= d_k([\eta], [\nu]). \end{aligned}$$

Consider the equicontinuous family of maps between the compact metric spaces  $V_{\mathcal{F}}(A_1)^{(1)}$  and  $V_{\mathcal{F}}(A'_{n_k})^{(1+s)}$  given by the family of restrictions

$$\{\hat{\psi}_k|_{V_{\mathcal{F}}(A_1)^{(1)}}\}_{k=1}^{\infty}.$$

By the Ascoli-Arzelà theorem there is a uniformly convergent subsequence,  $\hat{\psi}_{k,1}$  say. Consider next the restrictions

$$\hat{\psi}_{k,1}|_{(V_{\mathcal{F}}(A_2)^{(1)} \cup V_{\mathcal{F}}(A_2)^{(2)})} \rightarrow V_{\mathcal{F}}(A'_{n_2})^{(1+s)} \cup V_{\mathcal{F}}(A'_{n_2})^{(2+s)}$$

and similarly obtain a uniformly convergent subsequence  $\hat{\psi}_{k,2}$ . Continue in this way and select a diagonal subsequence  $\hat{\psi}_{m_k}$  which converges uniformly on  $V_{\mathcal{F}}(A_j)^{(t)}$  for all  $t, j$ . The limit map,  $\Gamma_0$  say, inherits the properties of the maps  $\hat{\psi}_k$  in being a semiring homomorphism which respects the right action of  $V_{\mathcal{F}}$ . Furthermore  $\Gamma_0$  is contractive and graded and

determines a scale respecting commuting diagram

$$\begin{array}{ccc} V_{\mathcal{F}}(A_j) & \longrightarrow & V_{\mathcal{F}}(A_{j+1}) \\ \downarrow \Gamma_0 & & \downarrow \Gamma_0 \\ V_{\mathcal{F}}(A_{n_j}) & \longrightarrow & V_{\mathcal{F}}(A'_{n_{j+1}}) \end{array}$$

where the horizontal maps are the given embeddings. It follows that  $\Gamma_0$  determines a contractive homomorphism of invariants

$$\Gamma : (V_{\mathcal{F}}(A), \Sigma(A), d) \rightarrow (V_{\mathcal{F}}(A'), \Sigma(A'), d').$$

One could appeal to the lifting arguments of Theorem 4.3 at this point to construct the desired injection but there is a direct shortcut which makes use of the commuting diagrams above. By uniqueness it follows that for any lifting  $\theta_j$  of the restriction

$$\Gamma_0 : V_{\mathcal{F}}(A_j) \rightarrow V_{\mathcal{F}}(A'_{n_j})$$

there is a lifting  $\theta_{j+1}$  of

$$\Gamma_0 : V_{\mathcal{F}}(A_{j+1}) \rightarrow V_{\mathcal{F}}(A'_{n_{j+1}})$$

which is an extension of  $\theta_j$ . Thus we may obtain a sequence  $\{\theta_j\}$  in this manner which determines the desired star extendible isomorphism  $\square$

**Corollary 6.5.** *There is a constant  $\delta > 0$  such that if  $A$  and  $A'$  are approximately finite nest algebras of diameter  $r$  and  $A \subseteq_{\delta} A'$  and  $A' \subseteq_{\delta} A$  then  $A$  and  $A'$  are star extendibly isomorphic.*

**Remark 6.6** We note that the following family of finitely acting operator algebras does not have the stability property.

Let  $\mathcal{E}$  be the family

$$\mathcal{E} = \{T_2^{max} \otimes M_n : n = 1, 2, \dots\} \cup \{E_n \otimes M_m : n, m = 1, 2, \dots\}$$

where  $T_2^{max}$  is the inflation algebra of matrices

$$x = [a] \oplus \begin{bmatrix} a & c \\ 0 & b \end{bmatrix} \oplus [b]$$

and  $E_n$  is the digraph algebra

$$E_n = \begin{bmatrix} D_n & S_n \\ & D_n \end{bmatrix}$$

where  $S_n$  is the space of matrices in  $M_n$  with zero diagonal. Consider the star-extendible algebra homomorphism

$$\phi_n : T_2^{max} \rightarrow C^*(E_n)$$

for which

$$\phi_n(x) = \begin{bmatrix} aI_n & cP_n \\ & bI_n \end{bmatrix}$$

where  $P_n$  is the rank one projection in  $M_n$  all of whose entries are  $1/n$ . Since  $P_n$  is close to  $S_n$  it is clear that

$$\phi_n(T_2^{max}) \subseteq_{\delta_n} E_n$$

where  $\delta_n \rightarrow 0$  as  $n \rightarrow \infty$ . On the other hand one can check that  $\|P_n - v_n\| \geq 1/3$  for any partial isometry  $v_n$  in  $S_n$ . (If  $v_n$  were close it would need to be of rank one and a rank one operator with zero diagonal is not close to  $P_n$ .) It follows that  $\|\phi - \psi\| \geq 1/3$  for any star extendible homomorphism  $\psi : T_2^{max} \rightarrow E_n$ . Thus the family  $\mathcal{E}$  and any family containing  $\mathcal{E}$  fails to be perturbationally stable.

This example suggests that the category AFA is indeed different from the category LIM.

**Remark 6.7.** It is plausible that the family of digraph algebras is perturbationally stable, although not uniformly so. The following basic question also appears to be open. Is a finitely acting operator algebra perturbationally stable? Choi and Davidson [5] have shown that close digraph algebras with common  $C^*$ -algebra are isomorphic which provides some evidence for the perturbational stability of digraph algebras. Also the 4-cycle algebras are known to give a stable family ([41]) and it seems likely that the  $2n$ -cycle algebras are also perturbationally stable.

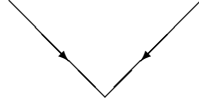
7. MATRICIAL  $V$ -ALGEBRAS

We now consider in detail the bipartite digraph algebra  $E$  in  $T_3$  and the family  $\mathcal{F} \subseteq \mathcal{F}_E$  of embeddings that preserve the strictly upper triangular ideal. This algebra  $E$  is an exceptional complete bipartite algebra in that the maps in  $\mathcal{F}$  are necessarily locally regular, although not necessarily regular. However the embeddings are 2-decomposable, as we see from Proposition 7.1 where we give complete invariants for inner conjugacy. This proposition leads to the identification of  $V_{\mathcal{F}}$  and the classification of algebras in  $\text{Lim}(\mathcal{F})$  and  $\text{Lim}(\tilde{\mathcal{F}})$ .

The algebra  $E$  consists of matrices of the form

$$\begin{bmatrix} a & x & y \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$$

and we write  $V$  for the bipartite graph whose proper edges are indicated by the  $V$ -shaped diagram



Let  $\varphi : E \otimes M_m \rightarrow E \otimes M_n$  belong to  $\mathcal{F}$ . Then we may write

$$\phi(e_{12}) = \begin{bmatrix} 0 & v_1 & v_2 \\ & 0 & \\ & & 0 \end{bmatrix}, \quad \phi(e_{13}) = \begin{bmatrix} 0 & w_1 & w_2 \\ & 0 & \\ & & 0 \end{bmatrix}.$$

Note that since  $\varphi$  is star-extendible the partial isometries  $\varphi(e_{12})$  and  $\varphi(e_{13})$  have block diagonal initial and final projections and so  $v_1, v_2, w_1, w_2$  are themselves partial isometries. In other words the map  $\varphi$  is necessarily locally regular.

**Proposition 7.1.** *Let  $\varphi, \psi : A(V) \otimes M_m \rightarrow A(V) \otimes M_n$  be maps with associated ordered quadruples  $\{v_1, v_2, w_1, w_2\}$ ,  $\{x_1, x_2, y_1, y_2\}$  respectively. Then  $\varphi$  and  $\psi$  are inner unitarily equivalent if and only if*

$$\text{rank}(v_i) = \text{rank}(x_i), \quad \text{rank}(w_i) = \text{rank}(y_i),$$



for  $i = 1, 2$ , and the positive operators  $w_1 w_1^* v_1 v_1^* w_1 w_1^*$ ,  $y_1 y_1^* x_1 x_1^* y_1 y_1^*$  are unitarily equivalent.

*Proof.* We begin the proof by putting the map  $\varphi$  into a standard form. First replace  $\varphi$  by a unitary conjugate  $(Adu) \circ \varphi$ , where the unitary  $u$  has the form

$$\begin{bmatrix} u_1 & 0 & 0 \\ & I & \\ & & I \end{bmatrix},$$

to arrange that  $w_1$  and  $w_2$  have standard orthogonal final projections of the form

$$w_1 w_1^* = f_{11} + \dots + f_{ss}, \quad w_2 w_2^* = f_{s+1,s+1} + \dots + f_{r,r}$$

where  $(f_{kl})$  are the matrix units of  $M_n$  and  $r$  is the multiplicity of the embedding. Next replace  $\varphi$  (the new  $\varphi$ ) by a further unitary conjugate, where the unitary has the form

$$\begin{bmatrix} I & 0 & 0 \\ & u_2 & \\ & & u_3 \end{bmatrix}$$

to arrange also that the initial projections of  $w_1$  and  $w_2$  are also standard orthogonal projections. In this way arrange the resulting standardisation of  $\phi(e_{13})$  to have partitioned matrix

$$\phi(e_{13}) = \left[ \begin{array}{c|cc|cc} 0 & 0 & I_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & I_2 \\ 0 & 0 & 0 & 0 & 0 \\ \hline & 0 & 0 & & \\ \hline & & & 0 & 0 \end{array} \right]$$

where  $I_1, I_2$  are sums of matrix units,  $\text{rank}(I_1) = s, \text{rank}(I_2) = t$  and  $s + t \leq n$ . Since  $\varphi(e_{12})$  has initial projection orthogonal to that of  $\varphi(e_{13})$  and since  $\varphi(e_{12})$  and  $\varphi(e_{13})$  have

the same final projection it follows that  $v_1$  and  $v_2$  have the induced partitioned matrices

$$v_1 = \left[ \begin{array}{c|c} v_{11} & 0 \\ \hline v_{12} & 0 \\ \hline 0 & 0 \end{array} \right], \quad v_2 = \left[ \begin{array}{c|c} v_{21} & 0 \\ \hline v_{22} & 0 \\ \hline 0 & 0 \end{array} \right].$$

Note that the projections  $P = v_1 v_1^*$  and  $Q = w_1 w_1^*$  are block diagonal, supported in the  $(1, 1)$  block only, with partitioned matrices

$$P = v_1 v_1^* = \left[ \begin{array}{c|c|c} R & S & 0 \\ \hline S^* & T & 0 \\ \hline 0 & 0 & 0 \end{array} \right], \quad Q = w_1 w_1^* = \left[ \begin{array}{c|c|c} I_1 & 0 & 0 \\ \hline 0 & 0 & 0 \\ \hline 0 & 0 & 0 \end{array} \right],$$

when  $R$  is an  $s \times s$  matrix and  $T$  is a  $t \times t$  matrix. To simplify notation, assume without loss of generality, that  $s+t = n$ , so that the last row and last column of the matrices above are not present. Assuming that  $\psi$  satisfies the rank equality conditions in the statement of the proposition we may also assume that  $\psi(e_{13})$  is standardised so that  $\psi(e_{13}) = \varphi(e_{13})$ . We claim that  $\psi$  and  $\varphi$  are inner conjugate if and only if the pair of projections  $\{P, Q\}$  in  $M_n$  is unitarily equivalent in  $M_n$  to the pair  $\{P', Q\}$ , when  $P' = x_1 x_1^*$ . In fact the necessity of this condition is elementary so assume that these pairs are unitarily equivalent, by a unitary  $Z$  in  $M_n$ . Since  $ZQZ^* = Q$  it follows that  $Z$  is block diagonal with respect to  $Q$ , so that

$$Z = \begin{bmatrix} Z_1 & 0 \\ 0 & Z_2 \end{bmatrix}.$$

Thus, conjugating  $\psi$  by the unitary

$$\begin{bmatrix} Z & 0 & 0 \\ & I_n & \\ & & I_n \end{bmatrix}$$

we obtain a unitarily equivalent embedding, also in standard form, with quadruple  $\{x_1, x_2, w_1, w_2\}$  satisfying  $v_1 v_1^* = x_1 x_1^*$  (as well as  $x_1^* x_1 = v_1^* v_1, x_2^* x_2 = v_2^* v_2$ ). Of necessity  $v_2 v_2^* = x_2 x_2^*$ , since, by star extendibility

$$v_1 v_1^* + v_2 v_2^* = \varphi(e_{12}) \varphi(e_{13})^* = \varphi(e_{11}) = \psi(e_{11}) = \psi(e_{12}) \psi(e_{12})^* = x_1 x_1^* + x_2 x_2^*.$$

Finally, observe that we may now conjugate  $\psi$  by a unitary of the form

$$\left[ \begin{array}{c|cc|cc} I_n & 0 & 0 & 0 & 0 \\ \hline & U_1 & & & \\ & & I_1 & & \\ \hline & & & U_2 & \\ & & & & I_2 \end{array} \right]$$

to obtain a map  $\psi'$  with quadruple  $\{x_1 U_1^*, x_2 U_2^*, w_1, w_2\}$ . Thus, with the choice  $U_1 = v_1^* x_1$ ,  $U_2 = v_2^* x_2$ ,  $\psi' = \varphi$  and the proof is complete.  $\square$

Let  $\varphi : A(V) \rightarrow A(V) \otimes M_n$  be as in the proof above, with  $\varphi(e_{13})$  standardised and consider again the decomposition of the projection

$$P = v_1 v_1^* = \left[ \begin{array}{c|c|c} R & S & 0 \\ \hline S^* & T & 0 \\ \hline 0 & 0 & 0 \end{array} \right]$$

induced by  $Q = w_1 w_1^*$  (with possibly zero rows and zero columns restored). It follows from the well-known spectral picture of a pair of projections that we can further decompose the projections  $Q = w_1 w_1^*$  and  $w_2 w_2^*$  so that  $P$  has the form

$$P = \left[ \begin{array}{c|c|c|c|c|c|c} I_3 & & & & & & \\ \hline & 0 & & & & & \\ & & C & & \sqrt{C(I-C)} & & \\ \hline \hline & & \sqrt{C(I-C)} & & I-C & & \\ & & & & & I_4 & \\ & & & & & & 0 \end{array} \right]$$

where, as always, the unmarked entries are zero matrices and where  $C$  and  $I - C$  are positive invertible contractions. (See Halmos [20].) We also have the degenerate possibility that  $C$  is absent.

Since  $v_1 v_1^* + v_2 v_2^* = w_1 w_1^* + w_2 w_2^*$  we deduce also that  $v_2 v_2^*$  has the complementary form

$$v_2 v_2^* = \left[ \begin{array}{c|c|c|c|c|c|c|c} 0 & & & & & & & \\ \hline & I_5 & & & & & & \\ \hline & & I - C & -\sqrt{C(I - C)} & & & & \\ \hline & & -\sqrt{C(I - C)} & C & & & & \\ \hline & & & & & 0 & & \\ \hline & & & & & & I_6 & \\ \hline & & & & & & & \end{array} \right]$$

An explicit indecomposable decomposition for  $\varphi$  can now be obtained.

Let  $\theta_i, 0 \leq i \leq 3$ , be the indecomposable regular embeddings from  $A(V)$  to  $A(V) \otimes M_2$  whose quadruples  $\{v_1^{(i)}, v_2^{(i)}, w_1^{(i)}, w_2^{(i)}\}$  have the rank distributions

$$\{1, 0, 0, 1\}, \{0, 1, 1, 0\}, \{0, 1, 0, 1\}, \{1, 0, 1, 0\}.$$

For example we can take  $\theta_2$  to be the map

$$\theta_2 : \begin{bmatrix} a & x & y \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} \rightarrow \begin{bmatrix} a & 0 & 0 & 0 & x & y \\ 0 & 0 & 0 & 0 & 0 & 0 \\ & & 0 & 0 & & \\ & & 0 & 0 & & \\ & & & b & 0 & \\ & & & 0 & c & \end{bmatrix}.$$

A moment's reflection on the pair of projections  $v_1 v_1^*, w_1 w_1^*$  in the standard form of  $\varphi$  above reveals that if those projections commute ( in which case the invertible contraction  $C$  is absent) then  $\varphi$  is regular and

$$[\varphi] = r_0[\theta_0] + r_1[\theta_1] + r_2[\theta_2] + r_4[\theta_4]$$

where  $r_0 = \text{rank}(I_4), r_1 = \text{rank}(I_3), r_2 = \text{rank}(I_6), r_3 = \text{rank}(I_3)$ . On the other hand if all these ranks are zero, then the spectral diagonalisation of the invertible strict contraction  $C$  gives rise to an indecomposable decomposition. The map  $\varphi$  is indecomposable in this case if  $C$  has rank one, corresponding to a scalar  $t$  in  $(0, 1)$ . We write  $\varphi_C$  and  $\varphi_t$  for the maps in these cases. Explicitly, for a positive invertible strict contraction  $C$  of rank  $n$  and

for  $m \geq 1$  we write  $id_m \otimes \varphi_C$  for the map from  $M_m \otimes A(V)$  to  $M_m \otimes A(V) \otimes M_{2n}$  given by

$$\phi_C \left( \begin{bmatrix} a & x & y \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} \right) = \left[ \begin{array}{c|cc|cc} a \otimes I_n & & & & & \\ & x \otimes \sqrt{C} & y \otimes I_n & x \otimes -\sqrt{I-C} & 0 & \\ \hline & a \otimes I_n & x \otimes \sqrt{I-C} & 0 & x \otimes \sqrt{C} & y \otimes I_n \\ \hline & & b \otimes I_n & & & \\ & & & c \otimes I_n & & \\ \hline & & & & b \otimes I_n & \\ & & & & & c \otimes I_n \end{array} \right].$$

This of course is the map induced by the map  $\varphi_C : A(V) \rightarrow A(V) \otimes M_{2n}$ , and the classes  $[id_m \otimes \varphi_C], [\varphi_C]$  in  $V_{\mathcal{F}}$  coincide.

The next proposition is the key to exposing the semiring structure of  $V_{\mathcal{F}}$ . In view of the discussion above the assertion follows also from the Krull Schmidt Theorem of Section 3 and the special case of one dimensional operators  $C, D$ . However the calculation required for this scalar case is no simpler than the calculation below.

**Proposition 7.2.** *Let  $C, D$  be positive invertible strict contractions in  $M_n$  and  $M_m$  respectively. In the semiring  $V_{\mathcal{F}}$  we have  $[\varphi_n][\varphi_C] = [\varphi_T]$  where  $T$  is a positive operator in  $M_{2nm}$  given by*

$$T = I_2 \otimes (I_{nm} - (I_n - C) \otimes (I_n - D)).$$

*Proof.* Let  $\varphi_C$  have quadruple  $\{v_1, v_2, w_1, w_2\}$  as before, let  $\varphi_D$  have quadruple  $\{v'_1, v'_2, w'_2, w'_2\}$  and consider the composition  $(id \otimes \varphi_D) \circ \varphi_C$ . The images of  $e_{12}$  and  $e_{13}$  give rise to the quadruple  $\{V_1, V_2, W_1, W_2\}$  for this composition.

Note that

$$v_1 = 1 \otimes \begin{bmatrix} \sqrt{C} & 0 \\ \sqrt{I-C} & 0 \end{bmatrix}, \quad v_2 = 1 \otimes \begin{bmatrix} -\sqrt{I-C} & 0 \\ \sqrt{C} & 0 \end{bmatrix},$$

$$W_1 = 1 \otimes P, \quad w_2 = 1 \otimes P^\perp,$$

where

$$P_m = \begin{bmatrix} I_m & 0 \\ 0 & 0 \end{bmatrix},$$

with a similar formula for  $v'_1, v'_2, w'_1, w'_2$ . Thus we may compute

$$\begin{aligned} V_1 &= v_1 \otimes v'_1 + v_2 \otimes w'_1, \\ W_1 &= w_1 \otimes v'_1 + w_2 \otimes w'_1. \end{aligned}$$

By orthogonality

$$\begin{aligned} V_1 V_1^* &= v_1 v_1^* \otimes v'_1 v'^{*}_1 + v_2 v_2^* \otimes w'_1 w'^{*}_1, \\ &= P_C \otimes P_D + P_C^\perp \otimes P_m, \\ W_1 W_1^* &= w_1 w_1^* \otimes v'_1 v'^{*}_1 + w_2 w_2^* \otimes w'_1 w'^{*}_1, \\ &= P_n \otimes P_D + P_n^\perp \otimes P_m. \end{aligned}$$

Thus  $W_1 W_1^* V_1 V_1^* W_1 W_1^*$  is equal to

$$\begin{aligned} &(P_n \otimes P_D + P_n^\perp \otimes P_m)(P_C \otimes P_D + P_C^\perp \otimes P_m)(P_n \otimes P_D + P_n^\perp \otimes P_m) \\ &= (P_n P_C \otimes P_D + P_n^\perp P_C^\perp \otimes P_D P + P_n^\perp P_C \otimes P_n P_D + P_n^\perp P_C^\perp \otimes P_m)(P_n \otimes P_D + P_n^\perp \otimes P_m) \\ &= P_n P_C P_n \otimes P_D + P_n P_C^\perp P_n \otimes P_D P_m P_D + (P_n^\perp P_C P_n \otimes P_m P_D + P_n^\perp P_C^\perp P_n \otimes P_m P_D) + \\ &\quad (P_n P_C P_n^\perp \otimes P_D P_m + P_n P_C^\perp P_n^\perp \otimes P_D P_m) + P_n^\perp P_C P_n^\perp \otimes P_m P_D P_m + P_n^\perp P_C^\perp P_n^\perp \otimes P_m. \end{aligned}$$

Note that each of the bracketed terms vanishes and so we obtain  $W_1 W_1^* V_1 V_1^* W_1 W_1^*$  as the sum of the three positive operators

$$\begin{aligned} &\left( \begin{bmatrix} C & 0 \\ 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} D & \sqrt{D(I-D)} \\ \sqrt{D(I-D)} & I-D \end{bmatrix} + \begin{bmatrix} I_n - C & 0 \\ 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} D & \sqrt{D(I-D)} \\ \sqrt{D(I-D)} & I-D \end{bmatrix} \right) D \\ &\quad + \left( \begin{bmatrix} 0 & 0 \\ 0 & I_n - C \end{bmatrix} \otimes \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} \right) + \left( \begin{bmatrix} 0 & 0 \\ 0 & C \end{bmatrix} \otimes \begin{bmatrix} I_m & 0 \\ 0 & 0 \end{bmatrix} \right). \end{aligned}$$

The first operator has nonzero spectral distribution

$$\{t + (1 - t)s : t \in \sigma(C), s \in \sigma(D)\}$$

with appropriate multiplicity whilst the sum of the second two operators which is orthogonal to the first has nonzero spectral distribution

$$\{(1 - t)s + t : t \in \sigma(C), s \in \sigma(D)\}$$

which is the same distribution. Appealing now to Proposition 7.2 the proof is complete.  $\square$

We can now identify  $V_{\mathcal{F}}$ .

Define the maps  $\phi_t : A(V) \rightarrow A(V) \otimes M_2$  for the values  $t = 0, 1$ , using the same specification as before and note that in  $V_{\mathcal{F}}$  we have

$$[\phi_0] = [\theta_0] + [\theta_1], [\phi_1] = [\theta_2] + [\theta_3]$$

and that  $\{[\phi_t] : 0 \leq t \leq 1\}$  is a homeomorph of the unit interval. Let  $\mathcal{F}_0 \subseteq \mathcal{F}$  be the (algebraically) closed family generated by the irregular embeddings  $\phi_t$  for  $0 < t < 1$ , with associated metrized semiring  $V_{\mathcal{F}_0} \subseteq V_{\mathcal{F}}$ . Let  $H$  be a fixed separable Hilbert space. Let  $\mathcal{C}_n$  be the set of strictly positive invertible contractions of rank  $n$  and let  $(X_n, d)$  be the metric space where  $X_n = \mathcal{C}_n / \sim$  is the set of unitary orbits of elements of  $\mathcal{C}_n$  and where  $d([C], [D])$  is the distance between the unitary orbits;

$$d([C], [D]) = \inf_U \|C - UDU^*\|.$$

More explicitly, if  $\sigma(C) = \{\lambda_1, \dots, \lambda_n\}$ ,  $\sigma(D) = \{\mu_1 \dots \mu_n\}$  with repetitions reflecting spectral multiplicity, then the distance  $d([C], [D])$  is the spectral distance

$$d(\sigma(C), \sigma(D)) = \inf_{\pi} \max_i |\lambda_i - \mu_{\pi(i)}|.$$

Set  $X = \sqcup_{n=1}^{\infty} X_n$  with the natural induced metric  $d$  for which  $d(x_n, y_m) = 1$  if  $x_n \in X_n, y_m \in X_m$  and  $n \neq m$ . Define a (graded) semiring structure on  $X$  by defining  $[C] + [D] = [C \oplus D]$  (using any identification of  $H \oplus H$  with  $H$ ) and defining  $[C][D] = [C \otimes D \otimes I_2]$  (using any identification of  $H \otimes H \otimes \mathbb{C}^2$  with  $H$ ). Proposition 7.2 shows that the bicontinuous

map  $\alpha$  given by

$$\alpha : (V_{\mathcal{F}_0}, d) \rightarrow (X, d), \quad \alpha : [\phi_C] \rightarrow [I - C]$$

is a bicontinuous semiring isomorphism. The completion of the metric space  $(V_{\mathcal{F}_0}, d)$  is the semiring  $(V_{\mathcal{F}_1}, d)$  where  $\mathcal{F}_1$  is the closed family generated by  $\{[\phi_t] : t \in [0, 1]\}$ .

Finally note that for any class  $[\psi]$  in  $V_{\mathcal{F}}$  the products  $[\psi][\theta_2], [\psi][\theta_3], [\theta_2][\psi], [\theta_3][\psi]$  are 1-decomposable and so as a set  $V_{\mathcal{F}}$  decomposes as a direct sum

$$\mathbb{Z}_+[\theta_0] + \mathbb{Z}_+[\theta_1] + \mathbb{Z}_+[\theta_2] + \mathbb{Z}_+[\theta_3] + V_{\mathcal{F}_0}.$$

The semigroup structure has been determined and the topology is the natural one consistent with  $(V_{\mathcal{F}_0}, d)$ .

In view of the perturbational stability of  $A(V)$  and the identification of  $V_{\mathcal{F}}$ , the abstract classification theorem of Theorem 6.2 applies and the dimension module invariants are computable in specific cases. Given the computability of compositions of embeddings we can obtain the following more explicit theorem.

Let  $\{C_k\}$  be a sequence of positive contractions in  $M_{r_k}$ , let  $n_k = 2^k r_1 r_2 \dots r_k$  and let

$$\psi_k : A(V) \otimes M_{n_k} \rightarrow A(V) \otimes M_{n_k} \otimes M_{2r_k}$$

be the usual embedding with  $V_{A(V)}$  class  $[\psi_k] = [\phi_{C_k}]$ . Write  $A(\{C_k\})$  for the unital operator algebra  $\varinjlim (A_k, \psi_k)$ , where  $A_k = A(V) \otimes M_{n_k}$ .

**Definition 7.3.** Two sequences  $\{C_k\}, \{D_k\}$  are asymptotically equivalent if there exist  $\epsilon_k > 0$  with  $\sum \epsilon_k < \infty$ , sequences  $(n_k), (m_k)$  and positive contractions  $(X_k), (Y_k)$  such that for all  $k$ ,

$$d\left(\left(\bigotimes_{i=n_k}^{n_{k+1}} (I - C_i)\right), (I - X_k) \otimes (I - Y_k)\right) < \epsilon_k,$$

$$d\left(\left(\bigotimes_{i=m_k}^{m_{k+1}} (I - D_i)\right), (I - Y_k) \otimes (I - X_{k+1})\right) < \epsilon_k.$$

Here  $d$  denotes the unitary orbit distance.



**Theorem 7.4.** *The operator algebras  $A(\{C_k\})$  and  $A(\{D_k\})$  are star extendibly isomorphic if and only if  $\{C_k\}$  and  $\{D_k\}$  are asymptotically equivalent.*

*Proof.* In view of Proposition 7.2 if the sequences are asymptotically equivalent then one can construct an asymptotically commuting diagram of unital star-extendible embeddings. From this it follows that  $A(\{C_k\})$  and  $A(\{D_k\})$  are star extendibly isomorphic. On the other hand by the stability of  $V$ -algebras if  $A(\{C_k\})$  and  $A(\{D_k\})$  are star extendibly isomorphic then there is an approximately commuting diagram that implements this isomorphism. From this it follows that the sequences are asymptotically equivalent.  $\square$

Write  $A_C$  for the unital stationary limit algebra  $A(\{C_k\})$  where  $C_k = C$  for all  $k$ . It now follows readily that if zero is not an eigenvalue of  $C$  then  $A_C$  is a regular limit algebra. Also if  $C, D$  are positive contractions in  $M_2$  with  $\sigma(C) = \{0, t\}, \sigma(D) = \{0, s\}$  with  $0 < s, t < 1$  then  $A_C$  and  $A_D$  are irregular limit algebras which are isomorphic if and only if  $t = s$ .

## 8. INFLATION ALGEBRAS

We now classify the limit algebras whose building block algebras are the inflation algebras  $T_r^{max} \otimes M_n$  and whose embeddings are arbitrary star extendible embeddings.

**Definition 8.1.** Let  $A \subseteq M_n$  be a digraph algebra and let  $\eta_i : A \rightarrow M_{n_i}, 1 \leq i \leq r$ , be unital embeddings of the form  $\eta_i(a) = p_i a p_i$ , where  $p_i$  is a semi-invariant projection for  $A$ . Then the operator algebra  $A_\phi = \phi(A(G))$  in  $M_{n_1} \oplus \cdots \oplus M_{n_r}$  determined by the algebra homomorphism  $\phi = \eta_1 \oplus \cdots \oplus \eta_r$  is called a (regular) inflation algebra.

The following example indicates why we shall be concerned here with the case of regular inflation algebras rather than general inflation algebras defined by contractive representations. Let  $0 < t < 1$  and let  $E \subseteq M_3 \oplus M_3$  be the "irregular" inflation algebra consisting of the matrices

$$\begin{bmatrix} a & x & z \\ & b & y \\ & & c \end{bmatrix} \oplus \begin{bmatrix} a & tx & t^2 z \\ & b & ty \\ & & c \end{bmatrix}.$$

Then it can be checked that  $E$  is rigid in the sense that if  $\phi : E \rightarrow E \otimes M_n$  is star extendible and indecomposable then either  $[\phi] = [id]$  or the range of  $\phi$  is contained in the self-adjoint subalgebra of  $E \otimes M_n$ .

Recall that the semi-invariant projections  $p$  of a digraph algebra  $A$  are precisely those projections in the centre of  $A \cap A^*$  for which the correspondence  $a \rightarrow pap$  determines an algebra homomorphism. The numbers of such projections, which have the form  $p_1 - p_2$  with  $p_1, p_2$  invariant projections, is clearly finite. Since repetitions of the compression embeddings have no effect on the star-extendible isomorphism type it follows that each digraph algebra has finitely many nonzero regular inflatomorphisms in OA.

Write  $A^{max}$  for the inflation algebra of  $A$  in which all the irreducible compression embeddings appear. Thus  $T_3^{max}$  is the operator algebra of matrices of the form

$$\begin{bmatrix} a & x & z \\ 0 & b & y \\ 0 & 0 & c \end{bmatrix} \oplus \begin{bmatrix} a & x \\ 0 & b \end{bmatrix} \oplus \begin{bmatrix} b & y \\ 0 & c \end{bmatrix} \oplus [a] \oplus [b] \oplus [c]$$

and its generated  $C^*$ -algebra,

$$C^*(T_3^{max}) = M_3 \oplus M_2 \oplus M_2 \oplus \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}$$

has maximal linear dimension amongst all the inflation algebras of  $T_3$ .

We shall show that such maximal inflation algebras  $T_r^{max}$  are of finite embedding type and compute the embedding rank. To do this we consider the link between star-extendible embeddings  $A^{max} \rightarrow A^{max} \otimes M_N$  and nonstar-extendible embeddings  $A \rightarrow A \otimes M_N$  which are regular in the normaliser preservation sense or, equivalently, which are of compression type in the sense below.

Assume that the digraph of  $A$  is connected and antisymmetric, so that  $C^*(A) = M_n$  and  $A \cap A^*$  is a masa in  $M_n$ , which we take to be  $D_n$ , the diagonal algebra. The compression homomorphisms  $\eta_1, \dots, \eta_r$  indicated in Definition 8.1 also determine homomorphisms onto their ranges, and we shall use the same notation for these maps and write  $\eta_k : A(G) \rightarrow A(G_k)$  where  $G_k$  is the appropriate subgraph of  $G$ .

**Definition 8.2.** A *compression type homomorphism*

$$\psi : A(G) \rightarrow A(H) \otimes M_N$$

is an algebra homomorphism which is unitarily equivalent to a direct sum of *elementary compression type homomorphisms*,  $\psi_1, \dots, \psi_s$ , each of which is a composition  $\mu_k \circ \eta_k$ , for some  $k$ , where

$$A(G) \xrightarrow{\eta_k} A(G_k) \xrightarrow{\mu_k} A(H) \otimes M_N$$

and  $\mu_k$  is an algebra injection arising from an identification of the digraph  $H_k$  with a subgraph of  $G \times K_n$ . Here  $K_n$  is the complete directed graph on  $N$  vertices.

Up to inner unitary equivalence there are finitely many indecomposables in the family of compression type homomorphisms and these are precisely the irreducible elementary compression type homomorphisms. When  $H = G$  these indecomposables are labelled by the elements of the semigroup  $\text{Pend}(G)$  of partial endomorphisms  $\alpha : G \rightarrow G$  where the domain of  $G$  is a connected subgraph determined by an interval and where  $\alpha$  is a digraph homomorphism. In Laurie and Power [30] it was shown that the compression type homomorphisms are precisely the contractive algebra homomorphisms between digraph algebras which are regular with respect to some pair of masas. This has a more direct proof in the case of  $T_r$ -algebras which we leave to the reader. Using this we may obtain the following classification of maps between the maximal inflation algebras  $T_r^{\max}$ ,  $r = 1, 2, \dots$ .

**Theorem 8.3.** *Let  $\lambda : T_r \rightarrow T_r^{\max}$ ,  $\kappa : T_s \rightarrow T_s^{\max}$  be the canonical (nonstar-extendible) completely isometric isomorphisms. Let  $\phi : T_r^{\max} \rightarrow T_s^{\max}$  be an algebra homomorphism and let  $\psi : T_r \rightarrow T_s$  be the algebra homomorphism  $\kappa^{-1} \circ \phi \circ \lambda$ . Then the following statements are equivalent.*

- (i)  $\phi$  is star-extendible.
- (ii)  $\psi$  is of compression type.
- (iii)  $\psi$  is a regular contractive homomorphism.

*Proof.* In the proof we will indicate the set of elementary compression type maps for  $T_r$  and  $T_s$  by  $\{\eta_i\}$  and  $\{\eta'_j\}$  respectively.

Suppose first that  $\psi$  is the elementary compression type homomorphism  $\mu \circ \eta$  where

$$T_r \xrightarrow{\eta} T_t \xrightarrow{\mu} T_s$$

where  $\eta$  is a compression type embedding determined by an interval projection of  $T_r$  of rank  $t$  and where  $\mu$  is a multiplicity one star-extendible injection mapping matrix units to matrix units. We wish to obtain a  $C^*$ -algebra extension

$$\tilde{\phi} : C^*(T_r^{\max}) \rightarrow C^*(T_s^{\max})$$

for the algebra homomorphism  $\phi = \kappa \circ \psi \circ \lambda^{-1}$ . Define first the restriction  $\tilde{\phi}_{res} = \tilde{\phi}|_{\eta(T_r)}$ , where  $\eta(T_r)$  now denotes the summand of  $T_r^{max}$  corresponding to  $\eta \in \{\eta_i\}$ , to be the map  $\mu$ , viewed as an algebra homomorphism from the summand  $\eta(T_r)$  to the largest summand  $\eta'_{id}(T_s)$  of  $T_s^{max}$ , where  $\eta'_{id} = \text{id}$ . Since  $\mu$  is star-extendible so too is this partial embedding  $\tilde{\phi}|_{\eta(T_r)}$ , with extension given by

$$\eta(M_r) \xrightarrow{\tilde{\mu}} M_s$$

where  $\tilde{\mu}$  is the star extension of  $\mu$ .

We now want to fully define  $\tilde{\phi}$  on all the other summands of  $C^*(T_r^{max})$  so that  $\tilde{\phi}$  is a  $C^*$ -algebra homomorphism and  $\tilde{\phi}(a) = \kappa \circ \psi \circ \lambda^{-1}$  for  $a$  in  $T_r^{max}$ . In view of the definition of  $T_s^{max}$ , a matrix  $b$  in  $T_s^{max}$  is determined by its largest summand, that is, by the  $s \times s$  matrix summand  $\eta'_{id}(T_s)$ . Thus the element  $\kappa \circ \psi \circ \lambda^{-1}(a)$  is a direct sum of various compressions  $\eta'_j(\mu(a))$  of the  $s \times s$  matrix  $\mu(a)$ . The key point to note is that each such compression of  $\mu(a)$ , which is determined by an interval of  $T_s$ , can be viewed as the image of a summand  $\eta_{ij}(a)$  of  $a$  under a star-extendible map. Let us denote this star-extension as  $\tilde{\phi}_j$ ; it is a multiplicity one  $C^*$ -algebra homomorphism from  $\eta_{ij}(M_r)$  to  $\eta'_j(M_s)$ . (If  $\eta'_j = \eta'_{id}$ , the largest compression, then  $\tilde{\phi}_i = \tilde{\phi}_{res}$ .) The map  $\tilde{\phi} = \sum_i \oplus \tilde{\phi}_i$  is the required extension.

We have shown that for any elementary compression type embedding  $\psi : T_r \rightarrow T_s$  the induced algebra homomorphism  $\phi : T_r^{max} \rightarrow T_s^{max}$  is star-extendible. It now follows that (ii) is implied by (i).

Consider now a star-extendible algebra homomorphism  $\phi : T_r^{max} \rightarrow T_s^{max}$ . This determines a contractive algebra homomorphism  $\psi : T_r \rightarrow T_s$  which may be viewed as the composition

$$T_r \xrightarrow{\lambda} T_r^{max} \xrightarrow{\phi} T_s^{max} \xrightarrow{\pi} T_s$$

where  $\pi$  is the restriction map for the maximal dimension summand of  $C^*(T_s^{max})$ .

Let  $v \in T_r^{max}$  be a partial isometry. Then it is straightforward to see that  $v$  is a unimodular sum of matrix units and that  $v = \lambda(u)$  where  $u$  is a unimodular sum of matrix units. Indeed, the maximal matrix summand of  $v$  is the matrix  $u$  which is a partial isometry along with each compression summand. From this it follows that  $u$  is a regular partial isometry in  $T_r$ .

Since  $\phi$  maps partial isometries to partial isometries, being star-extendible, it follows that  $\psi(v)$  is a regular partial isometry in  $T_s$ . Thus we conclude that  $\psi$  is a contractive regular homomorphism from  $T_r$  to  $T_s$ , and that (iii) holds.  $\square$

**Theorem 8.4.** *Let  $\lambda : T_r \rightarrow T_r^{max}$ ,  $\kappa \otimes id : T_s \otimes M_n \rightarrow T_s^{max} \otimes M_n$  be the canonical (nonstar-extendible) completely isometric isomorphisms. Let  $\phi : T_r^{max} \rightarrow T_s^{max} \otimes M_n$  be an algebra homomorphism and let  $\psi : T_r \otimes M_n \rightarrow T_s \otimes M_n$  be the algebra homomorphism  $(\kappa \otimes id)^{-1} \circ \phi \circ \lambda$ . Then the following statements are equivalent.*

- (i)  $\phi$  is star-extendible.
- (ii)  $\psi$  is of compression type.
- (iii)  $\psi$  is a regular contractive homomorphism.

A proof may be given along the lines above but for the final step. It is not apparent (as it is in the triangular case) that the locally regular algebra homomorphism  $\psi$  is necessarily a regular contractive homomorphism. That this is true for star-extendible maps was obtained recently in Hopenwasser and Power [25] and the proof we give below is a small variation of that one. For general digraph algebras locally regular star extendible maps need not be regular and so the maximal triangular structure is required in the proof.

**Theorem 8.5.** *Let  $\psi : T_r \rightarrow T_s \otimes M_n$  be a contractive algebra homomorphism. Then the following conditions are equivalent.*

- (i)  $\psi$  is locally regular in the sense that the image of each matrix unit is a regular partial isometry.
- (ii)  $\psi$  maps regular partial isometries to regular partial isometries.
- (iii)  $\psi$  is of compression type
- (iv)  $\psi$  is a regular contractive homomorphism.

*Proof.* The equivalence of (i) and (ii) is elementary and we have already noted the equivalence of (iii) and (iv). Plainly (iv) implies (i).

Assume condition (i). Let  $v^{ij} = \psi(e_{ij})$  have  $s \times s$  block decomposition  $(v_{pq}^{ij})$ , with  $1 \leq p, q \leq s, 1 \leq i, j \leq r$ . By assumption each  $v_{pq}^{ij}$  is a partial isometry. Consider a product  $v^{ij}v^{jk}$ . The  $(1, 1)$  block entry is given by the sum

$$v_{11}^{ij}v_{11}^{jk} + v_{12}^{ij}v_{21}^{jk} + \cdots + v_{1r}^{ij}v_{r1}^{jk}.$$

Since  $v^{ij}$  is regular, the partial isometries  $v_{11}^{ij}, \dots, v_{1r}^{ij}$  have orthogonal range projections and so the operators of the sum have orthogonal range projections. For similar reasons the domain projections are pairwise orthogonal. Since, by hypothesis, the product  $v^{ij}v^{jk}$  is a regular partial isometry, it follows that the sum above is a partial isometry, and therefore, by the orthogonality of domain and range projections, each of the individual products

$$v_{11}^{ij}v_{11}^{jk}, v_{12}^{ij}v_{21}^{jk}, \dots, v_{1r}^{ij}v_{r1}^{jk}$$

is a partial isometry.

Since, for example,  $v_{11}^{ij}v_{11}^{jk}$  is a partial isometry it follows that the range projection of  $v_{11}^{jk}$  commutes with the domain projection of  $v_{11}^{ij}$ . Regarding the entry operators  $v_{st}^{ij}$  as identified with operators in  $M_s \otimes M_n$ , it follows, by considering other block entries, that for all  $i, j, k, l, s, t, u, v$  the range projection of  $v_{st}^{ij}$  commutes with the domain projection of  $v_{uv}^{kl}$ . Note also that the domain projections and the range projections commute amongst themselves. Furthermore it is clear that these projections commute with the projections in the centre of the block diagonal subalgebra of  $M_s \otimes M_n$ .

Let  $p_1$  be a rank one projection which is dominated by  $v_{11}^*v_{11}$ . By the commutativity there is a maximal family  $p_1, \dots, p_t$  of rank one projections satisfying  $p_i v^{i,i+1} = p_i v^{i,i+1} p_{i+1}$ . The projection  $p_1 + \dots + p_t$  commutes with  $\psi(T_s)$  and determines an elementary compression type embedding summand of  $\psi$ . Now (iii) follows from induction.  $\square$

To identify  $V_E$  for  $E = T_r^{max}$  we need the following combinatorial facts.

Let  $[r]$  denote the totally ordered set  $\{1, 2, \dots, r\}$  and let  $[r, t]$  denote the number of order preserving functions  $f : [t] \rightarrow [r]$ , for  $t, r$  in  $\mathbb{N}$ . Since the  $[r+1, t]$  order preserving functions  $g$  from  $[t]$  to  $[r+1]$  are partitioned into sets according to the cardinality of  $g^{-1}(1)$  we have the recurrence identity

$$[r+1, t] = [r, t] + [r, t-1] + \dots + [r, 1] + 1.$$

Thus

$$[r+1, t+1] = [r, t+1] + [r+1, t]$$

from which it follows that  $[n, t]$  is the binomial coefficient  $\binom{n+t-1}{t}$ . In particular, if  $G_r$  is the digraph for  $T_r$  then we see that  $\text{End}(G_r)$  is a semigroup of cardinality  $\binom{2r-1}{r}$ , since it is identifiable with the semigroup of order preserving functions  $g : [r] \rightarrow [r]$ .

Consider now the semigroup of partially defined order preserving functions  $h : [t] \rightarrow [r]$  whose domains are intervals. Write  $< r, t >$  for the cardinality of this set of functions. The set is partitioned by the cardinality of the domain of  $h$ , that is, by the numbers  $1, 2, \dots, t$  and this leads to the identity

$$< r, t > = r[r, 1] + (r-1)[r, 2] + \dots + [r, t].$$

Indeed, there are  $r$  possible singleton domains,  $r-1$  domains of cardinality two, and so on. Thus

$$< r, t > = r \binom{r}{1} + (r-1) \binom{r+1}{2} + \dots + \binom{2r-1}{r}.$$

However, we have the binomial coefficient identity

$$\binom{2r+1}{r+1} = (r+1) \binom{r}{0} + r \binom{r}{1} + (r-1) \binom{r+1}{2} + \dots + \binom{2r-1}{r},$$

(which may be obtained by counting paths in Pascal's triangle) and so

$$< r, r > = \binom{2r+1}{r+1} - (r+1).$$

Note that the functions  $h$  label the classes of indecomposable compression type embeddings  $\eta : T_r \rightarrow T_r \otimes M_n$  (with  $n \geq t$  say) and by Theorem 8.5 these in turn correspond to the equivalence classes of indecomposable maps  $\phi : T_r^{max} \rightarrow T_r^{max} \otimes M_n$  (for large enough  $n$ ).

Let  $\mathcal{P}_r$  be the chain poset  $\{1, \dots, r\}$ . Define  $\text{Pend}(\mathcal{P}_r)$  to be the semigroup of partially defined endomorphisms (monotone maps) from  $\mathcal{P}_r$  to  $\mathcal{P}_r$  whose domains are intervals of  $\mathcal{P}_r$ . Thus the cardinality of  $\text{Pend}(\mathcal{P}_r)$  is  $< r, r >$  which is the embedding rank  $d(T_r^{max})$ . (It is curious that the embedding rank sequence  $d_r = d(T_r^{max})$ , for  $r = 1, 2, \dots$ , namely,

$$1, 7, 31, 121, 456, 1709, 6427, 24301, \dots$$

gives a new addition to the On-Line Encyclopedia of Integer Sequences, [46].)

**Theorem 8.6.** *Let  $E = T_r^{max}$ . Then the semiring  $V_E$  is isomorphic to the semiring  $\mathbb{Z}_+[\text{Pend}(\mathcal{P}_r)]$  (with the discrete metric). As an additive abelian semigroup  $V_E = \mathbb{Z}_+^{d(E)}$  where the embedding rank is*

$$d(E) = \binom{2r+1}{r+1} - (r+1).$$

*Proof.* The first part of the theorem follows from the arguments above which show that the indecomposable maps  $T_r^{max} \rightarrow T_r^{max} \otimes M_n$ , ( $n > r$ ) are labelled by the partially defined order preserving functions  $g : \{1, \dots, r\} \rightarrow \{1, \dots, r\}$  whose domains are intervals. The second assertion follows from the combinatorial discussion.  $\square$

The next theorem reduces the isomorphism problem for limits of  $T_r^{max}$ -algebras to the structure of the embedding semigroup. For example, one can now compute, in principle (and in practice with computer aid) all the stationary  $T_r^{max}$ -algebra limit algebras  $A_\phi$  determined by embeddings  $\phi$  in  $V_{T_r^{max}}$  of a particular multiplicity of low order.

**Theorem 8.7.** *Let  $A$  and  $A'$  be operator algebras in  $\text{Lim}(\mathcal{E})$  where  $\mathcal{E}$  is the family of  $T_r^{max}$ -algebras. Let*

$$V(A) = \varinjlim (\mathbb{Z}_+^{d_r}, \hat{\phi}_k), \quad V(A') = \varinjlim (\mathbb{Z}_+^{d_r}, \hat{\phi}'_k)$$

*be the dimension  $V_{T_r^{max}}$ -modules of  $A$  and  $A'$ . Then  $A$  and  $A'$  are stably star extendibly isomorphic if and only if  $V(A)$  and  $V(A')$  are isomorphic, and are star extendibly isomorphic if and only if  $V(A)$  and  $V(A')$  are isomorphic by a scale preserving isomorphism.*

*Proof.* The sufficiency direction follows from Theorem 5.2 and Theorem 8.6. The necessity of the condition, that is, the fact that  $V(A)$  is an invariant, will follow from Theorem 6.2 once we show the stability of  $T_r^{max}$ . However this follows readily from the perturbational stability of  $T_r$ .

Let  $\alpha : \rightarrow C^*(T_r^{max} \otimes M_n)$  be star extendible and suppose that

$$\alpha(T_r^{max}) \subseteq_\delta T_r^{max} \otimes M_n.$$

Then consider the map  $\beta : T_r \rightarrow M_r \otimes M_n$  given by  $\beta = \pi \circ \alpha \circ \lambda$  where

$$\pi : C^*(T_r^{max} \otimes M_n) \rightarrow M_r \otimes M_n$$



is the projection onto the largest summand. Since  $\alpha$  is an almost inclusion it follows that

$$\beta(T_r) \subseteq_\delta T_r \otimes M_n.$$

By Haworth's theorem, for  $\delta$  sufficiently small  $\beta$  is close to a star extendible map  $\gamma : T_r \rightarrow T_r \otimes M_n$ . Now it follows that the map

$$(\lambda \otimes id) \circ \gamma \circ \lambda^{-1} : T_r^{max} \rightarrow T_r^{max} \otimes M_n$$

is close to  $\alpha$ . □

## 9. FUNCTORIALITY AND ISOCCLASSIC FAMILIES.

We now consider the classification problem for limit algebras determined by proper families of maps in  $\mathcal{F}_E$ .

Let  $\mathcal{F}$  be a closed family of maps, as in Section 2. Then  $\mathcal{F}$  gives rise to a number of categories (additive  $\mathbb{C}$ -categories) the most elementary of which is the category  $\text{Sys}(\mathcal{F})$  whose objects consist of direct systems

$$\mathcal{A} : A_1 \xrightarrow{\phi_1} A_2 \xrightarrow{\phi_2} A_3 \longrightarrow \dots$$

with  $\phi_k \in \mathcal{F}$  for all  $k$ , and whose morphisms are determined by commuting diagrams with maps from  $\mathcal{F}$ . In particular,  $\Phi : \mathcal{A} \rightarrow \mathcal{A}'$  is an isomorphism of  $\text{Sys}(\mathcal{F})$  if there exists a commuting diagram of maps

$$\begin{array}{ccccccc} A_{n_1} & \xrightarrow{\quad} & A_{n_2} & \xrightarrow{\quad} & A_{n_3} & \xrightarrow{\quad} & \dots \\ & \searrow & & \searrow & & \searrow & \\ & & A'_{m_1} & \xrightarrow{\quad} & A'_{m_2} & \xrightarrow{\quad} & \dots \end{array}$$

where the horizontal maps are compositions of the given embeddings for  $\mathcal{A}, \mathcal{A}'$  and the crossover maps lie in  $\mathcal{F}$ .

Note that the scaled dimension module  $V_{\mathcal{F}}(\mathcal{A})$  is an invariant for morphisms in  $\text{Sys}(\mathcal{F})$  and in view of Theorem 4.3 is a complete invariant. Thus the dimension module invariants resolve the isomorphism problem for  $\text{Sys}(\mathcal{F})$ .

Define the category  $\text{Alglim}(\mathcal{F})$  whose objects are the operator algebras obtained as algebraic direct limits  $A_0 = \varinjlim A_k$  of the systems  $\mathcal{A} = \{A_k, \phi_k\}$  of  $\text{Sys}(\mathcal{F})$ . The morphisms

of  $\text{Alglim}(\mathcal{F})$  are the star extendible algebra homomorphisms. The category  $\text{Lim}(\mathcal{F})$  of closed operator algebras we have already indicated and there are obvious functors

$$\text{Sys}(\mathcal{F}) \xrightarrow{F} \text{Alglim}(\mathcal{F}) \xrightarrow{G} \text{Lim}(\mathcal{F})$$

However it may not be clear, even for rather elementary closed families, whether or not  $F$  or  $G$  induces injections (and hence bijections) between isoclasses, that is, between the isomorphism equivalence classes of the objects of each category. (We leave aside here the further categorical questions arising from other morphisms such as algebraic and bicontinuous morphisms. Nevertheless see Donsig, Hudson and Katsoulis [12] for this consideration in the case of regular limit algebras.) In this connection we introduce the notion of a functorial family of maps.

**Definition 9.1.** Let  $\mathcal{F}$  be a closed family of maps. Then  $\mathcal{F}$  is said to be functorial if for any commuting diagram

$$\begin{array}{ccccccc} A_1 & \longrightarrow & A_2 & \longrightarrow & A_3 & \longrightarrow & \\ & \searrow & & \nearrow & & \searrow & \\ & & A'_1 & \longrightarrow & A'_2 & \longrightarrow & \end{array}$$

in which the horizontal maps belong to  $\mathcal{F}$  and the crossover maps are star extendible homomorphisms there are sequences  $(m_k), (n_k)$  such that for the induced diagram

$$\begin{array}{ccccccc} A_{n_1} & \longrightarrow & A_{n_2} & \longrightarrow & A_{n_3} & \longrightarrow & \\ & \searrow & & \nearrow & & \searrow & \\ & & A'_{m_1} & \longrightarrow & A'_{m_2} & \longrightarrow & \end{array}$$

the crossover maps belong to  $\mathcal{F}$ .

One property which clearly leads to functoriality is the following factorisation property.

**Definition 9.2.** Let  $\mathcal{F}$  be a closed family of maps. Then  $\mathcal{F}$  is said to have the factorisation property if whenever  $\alpha$  is a map of  $\mathcal{F}$  with a factorisation  $\alpha = \psi \circ \phi$  where the domains of  $\phi$  and  $\psi$  are in the family of domain algebras for the family  $\mathcal{F}$ , then  $\phi$  and  $\psi$  belong to  $\mathcal{F}$ .

Plainly the functor  $F$  induces an isoclass bijection if  $\mathcal{F}$  is functorial. In this case morphisms for  $\text{Alglim}(\mathcal{F})$  actually derive from morphisms of  $\text{Sys}(\mathcal{F})$ . However the converse does not hold; there do exist nonfunctorial families for which  $F$  induces an isoclass bijection.

The following terminology is convenient, particularly in the consideration of regular systems.

**Definition 9.3.** Let  $\mathcal{F}$  be a closed family of maps. Then  $\mathcal{F}$  is said to be an isoclassic family if the functor  $F$  is bijective.

Let  $\mathcal{F}_G^{\text{reg}}$  be the closed family of regular maps  $A(G) \otimes M_n \rightarrow A(G) \otimes M_m$  where  $A(G)$  is a digraph algebra. It is an interesting open question whether  $\mathcal{F}_G^{\text{reg}}$  is always an isoclassic family.

For the  $2n$ -cycle digraph  $D_{2n}$ , with  $n \geq 3$ , it was shown in Donsig and Power [15] that the family of rigid regular embeddings (those whose indecomposables derive from automorphisms of  $D_{2n}$ ) is a family with the factorisation property and so functorial and isoclassic. Also it is shown there that the arguments admit perturbations and that  $G$  as well as  $F$  gives an isomorphism of categories. Combining this with Theorem 4.3 we obtain the following alternative to the  $K_0H_1$  classification scheme of [15].

**Theorem 9.4.** *Let  $n \geq 3$  and let  $\mathcal{A}, \mathcal{A}'$  be direct systems of  $2n$ -cycle algebras where the embeddings belong to the family  $\mathcal{F}$  of maps of rigid type. Then the following statements are equivalent*

- (i)  $\mathcal{A}, \mathcal{A}'$  are isomorphic systems of  $\text{Sys}(\mathcal{F})$ .
- (ii)  $A_0 = \varinjlim \mathcal{A}, A'_0 = \varinjlim \mathcal{A}'$  are star extendibly isomorphic algebras.
- (iii)  $A = \varinjlim \mathcal{A}, A' = \varinjlim \mathcal{A}'$  are star extendibly isomorphic operator algebras.
- (iv) There is a scaled ordered group isomorphism

$$(G_{\mathcal{F}}(A), \Sigma_{\mathcal{F}}(A)) \rightarrow (G_{\mathcal{F}}(A'), \Sigma_{\mathcal{F}}(A'))$$

which respects the  $D_{2n}$ -action on the positive cones.

The case  $n = 2$  of 4-cycle algebras considered in Power [44] requires different methods because in contrast to  $n \geq 3$  star extendible homomorphisms need not be locally regular. The functor  $F$  is shown to biject isoclasses despite the lack of the functorial property and

the functor  $G$  is shown to biject isoclasses at least in the case of odd systems. (The even case requires a more detailed perturbational analysis.) In fact the 4-cycle algebra is more naturally viewed as one of the family of bipartite digraph algebras considered in Section 9.

It should be clear now that the two general problems indicated in the introduction must be addressed in order to formulate and analyse invariants for limit algebras. In particular we have the specific problem of determining semiring  $R_G$  which arises from the functorial completion of the family of 1-decomposable embeddings of a digraph algebra  $A(G)$ .

#### 10. FUNCTORIALITY OF REGULAR $T_3$ -ALGEBRA MAPS.

We now show that the family  $\mathcal{F}_{T_3}^{reg}$  is functorial and hence isoclassic and we obtain a dimension module classification of the algebraic direct limits. It seems quite plausible that a somewhat more general argument would show the corresponding facts for  $T_r$ -algebras with  $r \geq 4$ .

We say that a map  $\varphi : T_3 \otimes M_n \rightarrow T_3 \otimes M_m$  is  $T_2$ -degenerate if  $\varphi(1)$  is dominated by the sum of two atomic interval projections of the range algebra. Such a map is necessarily regular for the following reasons. Firstly the image of each matrix unit is a regular partial isometry in the sense that the block entries are partial isometries. (See the discussion of Example 3.5.) Secondly such locally regular maps between block upper triangular matrix algebras are necessarily regular. This is a special feature of upper triangular matrix algebras given in Theorem 8.5. Also we say that a  $T_3$ -algebra map  $\varphi$  is of  $T_2$ -character if each indecomposable summand of  $\varphi$  is  $T_2$ -degenerate. In particular the composition  $\varphi \circ \psi$  is also  $T_2$ -degenerate for an arbitrary map  $\psi_1$  and so is regular. Thus  $\mathcal{F}_{T_3}^{reg}$  is not a family with the factorisation property.

Consider now a direct system of  $T_3$ -algebras

$$A_1 \xrightarrow{\phi_1} A_2 \xrightarrow{\psi_1} A_3 \xrightarrow{\phi_2} A_4 \xrightarrow{\psi_2} \dots$$

where each map  $\varphi_k$  is an irregular embedding of  $T_3 \otimes M_{n_{2k}-1}$ , and where the restriction of  $\varphi_k$  to  $T_2 \otimes M_{n_k}$  is regular, for each  $k$ . Here  $T_2$  is identified with the subalgebra of  $T_3$  spanned by  $e_{11}, e_{12}, e_{22}$ . Assume also that the maps  $\psi_k$  are  $T_2$ -degenerate with  $\psi_k(1)$  contained in  $T_2 \otimes M_{n_{2k+1}}$ . Then this direct system determines a commuting diagram isomorphism between the regular systems  $\{A_{2k-1}, \psi_k \circ \varphi_k\}, \{A_{2k}, \varphi_k \circ \psi_{k-1}\}$ . Since the crossover maps  $\varphi_k$  are irregular this example shows that in general it is necessary to take proper subsystems

in order to establish functoriality. The key lemma for the proof is the following converse to this kind of irregular factorisation.

**Lemma 10.1.** *Let  $\varphi : A_1 \rightarrow A_2, \psi : A_2 \rightarrow A_3$  be maps between  $T_3$ -algebras and suppose that  $\psi \circ \varphi$  is regular and that  $\varphi$  is irregular. Then  $\psi$  is of  $T_2$ -character.*

*Proof.* We may assume that  $A_1 = T_3, A_2 = T_3 \otimes M_m, A_3 = T_3 \otimes M_n$ . First note that if  $\varphi$  is irregular then for at least one of the matrix units  $e$  of the triple  $e_{12}, e_{23}, e_{13}$  the partial isometry  $v = \varphi(e)$  has the block form

$$v = \begin{bmatrix} a & x & z \\ & b & y \\ & & c \end{bmatrix}$$

where the operator  $b$  is not a partial isometry. To see this we argue by contradiction and assume otherwise. Since  $v^*v$  and  $vv^*$  are block diagonal, the operators  $a$  and  $c$  are partial isometries. By assumption,  $b$  is a partial isometry and so it follows that the operator

$$\begin{bmatrix} x & z \\ 0 & y \end{bmatrix}$$

is a partial isometry and also has block diagonal initial and final projections. But this implies that  $x, y$  and  $z$  are partial isometries. We deduce then that each operator  $\varphi(e_{ij}), 1 \leq i \leq j \leq 3$ , is a regular partial isometry, which is to say that  $\varphi$  is a locally regular map. By our remarks above this implies that  $\varphi$  is regular, contrary to hypothesis.

Since the entry  $b$  of  $\varphi(e)$  is not a partial isometry it follows, by reasoning as in the last paragraph, that  $x, y$  and  $z$  are not partial isometries and in particular are nonzero operators. Without loss of generality assume that matrix units for  $A_3$  are chosen so the restriction of the map  $\psi$  to the self-adjoint subalgebra has the form

$$\psi : a \oplus b \oplus c \rightarrow ((a \otimes P_{11}) \oplus (b \otimes Q_{11}) \oplus (c \otimes R_{11})) \oplus \dots \oplus ((a \otimes P_{33}) \oplus (b \otimes Q_{33}) \oplus (c \otimes R_{33}))$$

where the projections  $P = P_{11} + P_{22} + P_{33}, Q = Q_{11} + Q_{22} + Q_{33}$  and  $R = R_{11} + R_{22} + R_{33}$  have the same rank. More precisely, we can remove the rows and columns of  $A_3$  corresponding to the projection  $(1_{A_3} - \psi(1_{A_2}))$  and obtain the (typical) image  $\psi(v)$  in the operator matrix

form

$$\left[ \begin{array}{ccc|ccc|ccc} a \otimes P_{11} & x \otimes X_{11} & z \otimes Z_{11} & 0 & x \otimes X_{12} & z \otimes Z_{12} & 0 & x \otimes X_{13} & z \otimes Z_{13} \\ 0 & b \otimes Q_{11} & y \otimes Y_{11} & 0 & 0 & y \otimes Y_{12} & 0 & 0 & y \otimes Y_{13} \\ 0 & 0 & c \otimes R_{11} & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline & & & a \otimes P_{22} & x \otimes X_{22} & z \otimes Z_{22} & 0 & x \otimes X_{23} & z \otimes Z_{23} \\ & & & 0 & b \otimes Q_{22} & y \otimes Y_{22} & 0 & 0 & y \otimes Y_{23} \\ & & & 0 & 0 & c \otimes R_{22} & 0 & 0 & 0 \\ \hline & & & & & & a \otimes P_{33} & x \otimes X_{33} & z \otimes Z_{33} \\ & & & & & & 0 & b \otimes Q_{33} & y \otimes Y_{33} \\ & & & & & & 0 & 0 & c \otimes R_{33} \end{array} \right].$$

We shall now show that the map  $\psi$  is locally regular. Since the composition  $\psi \circ \varphi$  is assumed to be regular the matrix above, for the element  $v = \varphi(e)$ , is a regular partial isometry. In particular the  $(2, 2)$  block entry and the  $(2, 3)$  block entry have orthogonal ranges and so

$$\begin{bmatrix} 0 & 0 & 0 \\ x^* \otimes X_{23}^* & 0 & 0 \\ z^* \otimes Z_{23}^* & y^* \otimes Y_{23}^* & 0 \end{bmatrix} \begin{bmatrix} a \otimes P_{22} & x \otimes X_{22} & z \otimes Z_{22} \\ 0 & b \otimes Q_{22} & y \otimes Y_{22} \\ 0 & 0 & c \otimes R_{22} \end{bmatrix} = 0.$$

Thus  $x^*x \otimes X_{23}^*X_{22} = 0$ , and so  $X_{23}^*X_{22} = 0$ . If  $p$  is a rank one projection in  $M_m$  then the partial isometry  $\psi(e_{12} \otimes p)$  has block diagonal initial and final projections and, after removal of block rows and columns of zeros, has the  $3 \times 3$  block matrix form  $(p \otimes X_{ij})$ , where  $X_{ij} = 0$  for  $i > j$ . Since  $X_{23}^*X_{22} = 0$  it follows from the block diagonality of the range projections that  $X_{22}$  and  $X_{23}$  are partial isometries. Reasoning as before it follows that  $\psi(e_{12} \otimes p)$  is a regular partial isometry.

Similarly, since the  $(2, 3)$  block and the  $(3, 3)$  block of  $\psi(v)$  have orthogonal initial projections it follows that

$$\begin{bmatrix} 0 & x \otimes X_{23} & z \otimes Z_{23} \\ 0 & 0 & y \otimes Y_{23} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a^* \otimes P_{33} & 0 & 0 \\ x^* \otimes X_{33}^* & b^* \otimes Q_{33} & 0 \\ z^* \otimes Z_{33}^* & y^* \otimes Y_{33}^* & c^* \otimes R_{33} \end{bmatrix} = 0.$$

In particular  $yy^* \otimes Y_{23}Y_{33}^* = 0$  and so it follows, as before, that  $\psi(e_{23} \otimes p)$  is a regular partial isometry.

If  $U$  and  $V$  are regular partial isometries with  $U^*U = VV^*$  then it need not be the case that the partial isometry  $UV$  is regular. Thus we need additional argument in order to see that  $\psi(e_{13} \otimes p)$  is a regular partial isometry. Returning once more to the regularity of  $\psi(v)$  and the orthogonality of the range projections of the (2,2) and the (2,3) block entries, we see that

$$z^*z \otimes Z_{23}^*Z_{23} + y^*y \otimes Y_{23}^*Y_{23} = 0.$$

Since  $Y_{23}^*Y_{23} = 0$ , by the regularity of the partial isometry  $\psi(e_{23})$  it follows, since  $z \neq 0$ , that  $Z_{23}^*Z_{23} = 0$ . Since  $\psi(e_{13})$  has block diagonal final projection this implies that  $Z_{22}$  and  $Z_{23}$  are partial isometries, and so, as before,  $\psi(e_{13})$  is a regular partial isometry. Thus, by our earlier remarks,  $\psi$  is regular.

Suppose now that  $\psi_1$  is an indecomposable summand of  $\psi$  which is necessarily of multiplicity one. By the Krull-Schmidt theorem, Theorem 3.4, indecomposable decompositions are unique and from this it follows that since  $\psi \circ \varphi$  is regular so too is  $\psi_1 \circ \varphi$ . If  $\psi_1$  is not  $T_2$ -degenerate then  $\psi(v)$  is not a regular partial isometry, contrary to the regularity of  $\psi_1 \circ \varphi$ . It follows then that  $\psi$  is of  $T_2$ -character. □

**Theorem 10.2.** *Let  $\mathcal{F}_{T_3}^{reg}$  be the family of regular embeddings between  $T_3$ -algebras. Then  $\mathcal{F}_{T_3}^{reg}$  and  $\tilde{\mathcal{F}}_{T_3}^{reg}$  are isoclassic families.*

*Proof.* We first note the immediate consequence of Lemma 7.4 that if  $\mathcal{A} = \{A_k, \alpha_k\}$  and  $\mathcal{A}' = \{A'_k, \beta_k\}$  are  $T_3$ -algebra systems for which none of the embeddings  $\alpha_k, \beta_k$  and their system compositions are of  $T_2$ -character then a commuting diagram isomorphism between  $\mathcal{A}$  and  $\mathcal{A}'$  is necessarily regular.

In general consider the commuting diagram

$$\begin{array}{ccccccc}
 A_{n_1} & \xrightarrow{\quad} & A_{n_2} & \xrightarrow{\quad} & A_{n_3} & \xrightarrow{\quad} & \\
 & \searrow \phi_1 & \nearrow \psi_1 & & \searrow \phi_2 & \nearrow \psi_2 & \\
 & & A'_{m_1} & \xrightarrow{\quad} & A'_{m_2} & \xrightarrow{\quad} & 
 \end{array}$$

with  $\phi_k, \psi_k$  star extendible for all  $k$ . Suppose moreover that infinitely many of the maps  $\phi_k$  are irregular. Replacing the systems by subsystems we may assume that all these maps are irregular. By the lemma all the maps  $\psi_k$  are of  $T_2$ -character. Since  $\phi_k \circ \psi_{k-1}$  is regular

it must be that the range of the regular map  $\psi_{k-1}$  does not meet those off-diagonal blocks which contain rank one matrix units  $e$  for which  $\phi_k(e)$  is an irregular partial isometry. This implies that  $\phi_k \circ \psi_{k-1} \circ \phi_{k-1}$  is locally regular. Indeed the range of  $\psi_{k-1}$  meets more blocks of  $A_k$  than does the range of (the regular map)  $\psi_{k-1} \circ \phi_k$ . Since the triple composition is locally regular it is regular and it now follows that there is a commuting subdiagram of regular maps. Thus  $\mathcal{F}_{T_3}^{reg}$  is functorial and hence an isoclassic family. The argument is the same for  $\tilde{\mathcal{F}}_{T_3}^{reg}$ .  $\square$

We can now see that the locally finite algebras determined by regular embeddings of  $T_3$ -algebras have well-defined dimension module invariants and moreover are classified by these invariants. The classification of the corresponding operator algebras requires an perturbational version of the last lemma.

**Corollary 10.3.** *Let  $A_0, A'_0$  belong to  $\text{Alglim}(\mathcal{F}_{T_3}^{reg})$  and let*

$$V(A_0) = \varinjlim (\mathbb{Z}_+^{10}, \hat{\phi}_k), \quad V(A'_0) = \varinjlim (\mathbb{Z}_+^{10}, \hat{\phi}'_k)$$

*be their dimension modules with right action from the semigroup  $V_{\mathcal{F}_{T_3}^{reg}} = \mathbb{Z}_+^{10}$  determined by their defining direct systems. Then  $A_0$  and  $A'_0$  are stably star extendibly isomorphic if and only if the dimension modules  $V(A_0)$  and  $V(A'_0)$  are isomorphic, and are star extendibly isomorphic if and only if  $V(A_0)$  and  $V(A'_0)$  are isomorphic by a scale preserving isomorphism.*

## 11. BIPARTITE DIGRAPHS AND NONFUNCTORIALITY.

We now show the nonfunctoriality of regular embeddings of complete bipartite digraph algebras. This is done by constructing pairs of high multiplicity indecomposable embeddings with compositions that are 1-decomposable. We also indicate connections with subfactors and positions of self-adjoint subalgebras of  $C^*$ -algebras.

Let  $G_{n,m}$  be the complete bipartite digraph whose digraph algebra is

$$A(G_{n,m}) = \begin{bmatrix} \mathbb{C}^n & M_{n,m} \\ 0 & \mathbb{C}^m \end{bmatrix}$$

where  $M_{n,m}$  is the  $\mathbb{C}^n - \mathbb{C}^m$ -bimodule of  $n \times m$  complex matrices. Also write  $G_n$  for  $G_{n,n}$ . In particular  $A(G_1) = T_2$ ,  $A(G_2)$  is the 4-cycle algebra and  $A(G_{1,2})$  is the  $V$ -algebra of Section 7.



Let  $\mathcal{F}_n$  be the family of regular unital embeddings

$$\phi : A(G_n) \otimes M_r \rightarrow A(G_n) \otimes M_s, \text{ for } r < s,$$

which preserve the 2 by 2 block structure. Thus  $\phi$  may be indicated as

$$\phi = \begin{bmatrix} \phi_1 & \phi_{12} \\ 0 & \phi_2 \end{bmatrix}$$

where  $\phi_1, \phi_2 : \mathbb{C}^n \otimes M_r \rightarrow \mathbb{C}^n \otimes M_s$  are C\*-algebra maps and

$$\phi_{12} : M_{n,m} \otimes M_r \rightarrow M_{n,m} \otimes M_s$$

is an appropriate bimodule map. We shall show that  $\mathcal{F}_n$  is not a functorial family. The key construction for this is to obtain an irregular factorisation of a regular embedding  $\theta : A(G_n) \rightarrow A(G_n) \otimes M_{n^2}$  which is the direct sum of  $n^2$  maps  $\theta_{ij}$  arising from automorphisms  $\sigma_{ij} = \sigma_1^i \times \sigma_2^j$  of  $G_n$  where  $\sigma_1, \sigma_2$  are cyclic shifts of the range and source vertices. In Donsig and Power [14] this was obtained for  $n = 2$  by a seemingly fortuitous ad hoc argument.

Let  $n \geq 2$  be an integer and let  $w$  be a primitive root of unity. Let  $U$  be the  $n \times n$  unitary matrix

$$U = (u_{ij}) = \left( \frac{w^{(i-1)(j-1)}}{\sqrt{n}} \right)$$

and let  $S$  be the cyclic forward shift in  $M_n$  for the standard basis.

Let  $(f_{ij})$  be the standard matrix unit system for  $M_n$  where  $M_n$  is viewed as the  $(1, 2)$  block subspace of  $A(G_n)$ . Define  $\phi : A(G_n) \rightarrow A(G_n) \otimes M_n$  to be the restriction of the unique C\*-algebra map  $\tilde{\phi}$  between the generated C\*-algebras for which

$$\tilde{\phi} : \begin{bmatrix} 0 & f_{ij} \\ 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & ((S^*)^{i-1} U S^{j-1}) \otimes e_{ij} \\ 0 & 0 \end{bmatrix}$$

where  $(e_{ij})$  is the standard matrix unit system for the tensor factor  $M_n$ . Since

$$\phi : \begin{bmatrix} 0 & f_{ij} \\ 0 & 0 \end{bmatrix} \rightarrow (AdZ) \circ \phi \left( \begin{bmatrix} 0 & U \otimes e_{ij} \\ 0 & 0 \end{bmatrix} \right)$$

with  $Z$  a  $2 \times 2$  block diagonal unitary operator in  $C^*(A(G_n) \otimes M_n)$  it is clear that there is such a  $C^*$ -algebra map. Note that  $\phi$  is not a regular embedding. For example, for  $n = 3$  the embeddings has multiplicity 3 and

$$\phi(f_{11}) = \frac{1}{\sqrt{3}} \left[ \begin{array}{ccc|ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & w & 0 & 0 & w^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & w^2 & 0 & 0 & w^4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right].$$

Since the block entries have norm  $\frac{1}{\sqrt{3}}$  the embedding is not regular.

Let  $\bar{U}$  be the complex conjugate of the matrix  $U$  and define

$$\psi : A(G_n) \otimes M_n \rightarrow (A(G_n) \otimes M_n) \otimes M_n$$

to be the unique star algebra homomorphism such that for  $x$  in the tensor factor,

$$\psi : \begin{bmatrix} 0 & f_{ij} \\ 0 & 0 \end{bmatrix} \otimes x \rightarrow \left( \begin{bmatrix} 0 & (S^*)^{i-1} U S^{j-1} \\ 0 & 0 \end{bmatrix} \otimes e_{ij} \right) \otimes x.$$

**Lemma 11.1.** *The map  $\psi \circ \phi$  is inner conjugate to the 1-decomposable embedding  $\theta$ .*

*Proof.* Note that

$$(S^*)^{i-1} U S^{j-1} = \left( \frac{w^{(k+i-1)(l+j-1)}}{\sqrt{n}} \right)_{k,l=1}^n$$

and that

$$(k+i-1)(l+j-1) - (s+k-1)(t+l-1) = (i-1)(j-1) - (s-1)(t-1) + k(j-t) + l(i-t).$$

Thus we have the following calculation.

$$\begin{aligned}
(\psi \circ \phi)_{12}(f_{ij}) &= \psi_{12}((S^*)^{i-1} U S^{j-1}) \otimes e_{ij} \\
&= \psi_{12}(\sqrt{n}^{-1} (\sum_{k,l} w^{(k+i-1)(l+j-1)} f_{k,l}) \otimes e_{ij}) \\
&= \sqrt{n}^{-1} \sum_{k,l} w^{(k+i-1)(l+j-1)} (\psi_{12}(f_{k,l} \otimes e_{ij})) \\
&= \sqrt{n}^{-1} \sum_{k,l} w^{(k+i-1)(l+j-1)} \sqrt{n}^{-1} (\sum_{s,t} w^{(s+k-1)(t+l-1)} f_{st} \otimes e_{kl} \otimes e_{ij}) \\
&= \sum_{s,t} f_{st} \otimes (\sum_{k,l} w^{(i-1)(j-1)-(s-1)(t-1)} (\frac{w^{j-t}{}^k (w^{i-s})^l}{n} e_{kl}) \otimes e_{ij} \\
&= \sum_{s,t} f_{st} \otimes (X_{st}^{ij}) \otimes e_{ij}
\end{aligned}$$

where  $X_{st}^{ij}$  is a unimodular multiple of the rank one partial isometry

$$Y_{st}^{ij} = \sum_{k,l} \frac{(w^{j-t})^k (w^{i-s})^l}{n} e_{kl}.$$

We now want to show that the composition  $\psi \circ \phi$  is not merely locally regular, which is what the above calculation shows, but that it is regular, that is, 1-decomposable. (A star extendible locally regular map not be regular, as we have seen.)

Let  $(g_1, g_2, \dots, g_n)$  in  $\mathbb{C}^n$  be the basis with

$$g_i = (w^i, w^{2i}, \dots, w^{ni}) / \sqrt{n},$$

so that

$$Y_{st}^{ij} = g_{j-t} \otimes \bar{g}_{i-s},$$

where  $g_1 \otimes \bar{g}_2$  indicates the rank one operator for which  $(g_1 \otimes \bar{g}_2)(h) = \langle h, g_2 \rangle g_1$ .

Observe that the embedding  $\eta : A(G_n) \rightarrow A(G_n) \otimes M_{n^2}$  for which

$$\eta_{12}(f_{ij}) = \sum_{s,t} (f_{s,t} \otimes g_{j-t, i-s} \otimes e_{ij})$$

is a regular star extendible embedding unitarily equivalent to a map  $\theta$  determined by two cyclic shifts as indicated above. Thus  $\psi \circ \phi$  is the composition  $\eta \circ \theta$  where

$$\eta : M_{2n} \rightarrow M_{2n} \otimes M_n \otimes M_n$$

is the linear Schur product map given by

$$\eta(f_{st} \otimes e_{kl} \otimes e_{ij}) = w^{(i-1)(j-1)-(s-1)(t-1)} f_{st} \otimes e_{kl} \otimes e_{ij}.$$

Although these unimodular coefficients do not form a cocycle, that is,  $\eta$  is not realisable as a diagonal unitary conjugation, the restriction of  $\eta$  to the span of the matrix units

$$\{f_{s,t} \otimes g_{j-t,i-s} \otimes e_{ij} : 1 \leq s, t \leq n, 1 \leq i, j \leq n\}$$

is a cocycle. This may be checked directly. Alternatively note that since  $\psi \circ \phi = \eta \circ \theta$ , the map  $\psi \circ \phi$  is the orthogonal direct sum of  $n^2$  star extendible embeddings  $\eta \circ \theta_{ij}$ . Since  $\psi \circ \phi$  is star extendible so too is each map  $\theta_{ij}$ . That  $\eta$  is diagonally implementable on the ranges of the multiplicity one maps  $\theta_{ij}$  follows from the fact that an isometric Schur product map on the bipartite graph is diagonally implementable. (In fact this property is shown to hold for any digraph algebra in Davidson and Power [9].) Since there is a diagonal partition of the identity operator which dominates the ranges of the maps  $\eta \circ \theta_{ij}$  it follows that  $\eta \circ \theta$  is diagonally conjugate to  $\theta$ , as desired.  $\square$

Let  $\mathcal{G}_n \subseteq \mathcal{F}_n$  denote the closed subfamily of regular maps whose indecomposables are the multiplicity one embeddings corresponding to the automorphisms of  $G_n$ . The arguments above show that  $\mathcal{F}_n$  and  $\mathcal{G}_n$  are closed families which do not satisfy the factorisation property. In fact these families are not even functorial.

**Theorem 11.2.** *For  $n = 2, 3, \dots$  the families  $\mathcal{F}_n$  and  $\mathcal{G}_n$  are not functorial.*

*Proof.* Let  $\phi_k = \phi \otimes \text{id}, \psi_k = \psi \otimes \text{id}$ , be the maps

$$\phi \otimes \text{id} : A(G_n) \otimes M_{n^2k} \rightarrow (A(G_n) \otimes M_n) \otimes M_{n^2k},$$

$$\psi \otimes \text{id} : A(G_n) \otimes M_{n^2k+1} \rightarrow (A(G_n) \otimes M_n) \otimes M_{n^2k+1}.$$

Then for all  $k$  the compositions  $\psi_k \circ \phi_k$  are regular by the last lemma. Also, since  $\bar{w}$  is also a primitive root of unity the lemma shows that the compositions  $\phi_{k+1} \circ \phi_k$  are regular. Thus the maps  $\phi_k, \psi_k$  provide a commuting diagram between two regular systems in  $\text{Sys}(\mathcal{G}_n)$  consisting of irregular maps. Moreover it is clear that the crossover maps of any subdiagram are necessarily irregular, and so the theorem is established.  $\square$

### Subalgebra positions.

In [44], we analysed irregular factorisations in the case of the 4-cycle algebra  $A(G_2)$  and showed that there is a converse to the construction above in the following sense. If  $\Phi$  is an irregular star-extendible isomorphism between the systems  $\mathcal{A}, \mathcal{A}'$  in  $\text{Sys}(\mathcal{G}_2)$  then necessarily  $\mathcal{A}$  and  $\mathcal{A}'$  are systems determined by compositions of embeddings of type  $\theta$ . In particular  $\mathcal{A}$  and  $\mathcal{A}'$  are regularly isomorphic by some other isomorphism  $\Psi$ . Because of this  $\mathcal{G}_2$  is an isoclassic family. Noting that  $\mathcal{G}_2$  is the family of rigid embeddings we deduce that the equivalence between (i),(ii) and (iv) in Theorem 7.4 also holds for the case  $n = 2$  of 4-cycle algebras.

Let  $\mathcal{G}_2^{UHF} \subseteq \mathcal{G}_2$  be the subfamily of unital systems  $\mathcal{A}$  for which the algebraic direct limit has the form

$$A_0 = \begin{bmatrix} D_0 & M_0 \\ & D_0 \end{bmatrix}$$

where  $D_0$  is a unital ultramatrix algebra. It was shown in [44] how the inclusion

$$D_0 \oplus D_0 \subseteq B_0 = C^*(A_0)$$

determines the pair  $A_0, A_0^*$  and therefore how the  $K_0H_1$  classification scheme of Donsig and Power [14] gives a classification scheme for these positions. In themselves each summand  $D_0$  has Jones index 2 in the corresponding corner algebra of  $B_0$  and these positions are unique, in analogy with (although more elementary than) Goldman's theorem for index 2 subfactors. Thus the invariants may be viewed as determining the relative position of index 2 subcorners in the superalgebra. By taking weak closures in the tracial representation one obtains unital inclusions  $R \oplus R \subseteq R$ , where  $R$  is the hyperfinite  $\text{II}_1$  factor (with common Cartan masa) and where, again, the summands have index 2 in the corners. In this case, as one would expect, almost all of the  $K_0H_1$  invariants evaporate. The residue turns out to be the  $H_0H_1$  coupling invariant for partial isometry homology. One can reinterpret the

weak closures in the language of subfactor theory in terms of an  $R - R$ -bimodule  ${}_R M_R^\alpha$  determined by a symmetry  $\alpha$ , in which case the  $H_0 H_1$  coupling invariant corresponds to Connes spectral invariant [6] for the symmetry.

We expect that similar techniques will show that for the other bipartite graphs the family  $\mathcal{G}_n$  is isoclassic and hence that one can similarly obtain complete dimension module invariants for the bipartite (algebraic) limit algebras with respect to these regular embeddings. Once again this leads to invariants for regular subalgebra positions  $D_0 \oplus D_0 \subseteq M_0$  (of higher Jones index) and connections with subfactor theory. However, there are more possibilities for irregular factorisations of 1-indecomposable embeddings and it becomes interesting to determine the appropriate subfactor setting. This connection should lead to information on the number of approximately inner equivalence classes of standard diagonals (cf Donsig Power [13]) in the bipartite limit algebras (for  $\mathcal{G}_n$ ) and may perhaps shed light on the longstanding problem of the automorphic uniqueness of standard diagonals in regular limit algebras.

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